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STRUCTURE OF THE FUNDAMENTAL SOLUTION OF CAUCHY PROBLEM FOR KOLMOGOROV SYSTEMS OF SECOND-ORDER

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Abstract. We study a structure of the fundamental solution of the Cauchy problem for a class of ultra parabolic equations with a finite number of groups of variables with degenerated parabolicity.

Keywords: Kolmogorov systems, fundamental solutions, degenerate parabolic equations.

1. INTRODUCTION

In this paper we investigate the fundamental solution to the Cauchy problem (FSCP) for a class of systems of Kolmogorov equations [1, 2] which are a natural generalization of diffusion equation with inertia.

The equations that generalize Kolmogorov equations have been studied in many papers, especially a detailed description of the theories of the diffusion equations with inertia is presented in [3-5]. The great interest to study the behavior of solutions of Cauchy problem and boundary problems for Kolmogorov equations caused their wide application in Financial Mathematics for calculating the price of Asian options and volatility characteristics [6, 7].

We consider the system of equations with arbitrary number of groups of variables for which the parabolicity is degenerated and research the structure of FSCP. I particular we obtained exact dependence and types of shifts on lines of levels for FSCP of systems and model equations.

2. Notations and Formulation of the Problem

Let n, n_0 be fixed natural numbers and $n_0 > 1, x \in R^{n_0}, (x,s) = \sum\limits_{j=1}^{n_0} x_j s_j, \quad x^* = (x_2, ..., x_{n_0})$. Consider the following system of equations of the form

$$\partial_t u_\nu(t,x) - \sum_{j=1}^{n_0-1} x_j\, \partial_{x_{j+1}}\ u_\nu(t,x) = \sum_{k=0}^{2} \sum_{r=1}^{n} a_k^{\nu r}(t,x) \partial^k_{x_1^k}\ u_r(t,x),\ \nu = \overline{1,n}.\ x \in \Pi_{(0,T]}, \tag{2.1}$$

where $\Pi_{(0,T]} = \{(t,x),\ \ t \in (0,T],\ \ T > 0,\ \ x \in R^{n_0}\}$.
Assume that the coefficients $a_k^{nur}(t,x)$ of the system are complex-valued functions such that

$$\partial_t\ \omega_\nu(t,x) = \sum_{k=0}^{2} \sum_{r=1}^{n} a_k^{\nu r}(t,x) \partial^k_{x_1^k}\ \omega_r(t,x),\ \nu = \overline{1,n} \tag{2.2}$$

system (2.2) is uniformly parabolic in Petrovsky means in $\Pi_{[0,T]}$ and $(x_2, ..., x_{n_0})$ are considered as parameters. For convenience, we write the system (2.1) in matrix form:

$$\partial_t u(t,x) - \sum_{j=1}^{n_0-1} x_j \partial_{x_{j+1}}\ u(t,x) = \sum_{k=0}^{2} a_k(t,x)\ \partial^k_{x_1^k}\ u(t,x).$$

Find the solution of system (2.1) which satisfies the initial condition

$$u(t,x)|_{t=\tau} = u_0(x),\ x \epsilon R^{n_0},\ 0 \le \tau < t \le T, \tag{2.3}$$

where τ is a given number and $u_0 = col(u_{01},(x), ..., u_{0n},(x))$ is a given matrix column.

3. The Solution of Cauchy Problem for Systems with Constant Coefficients

Let us consider Cauchy problem for system (2.1) in which coefficients a_2^{nur} are constants $a_1^{\nu r} \equiv 0,\ a_0^{\nu r} = 0,\ \nu = \overline{1,n},\ r = \overline{1,n}$.

$$\partial_t u_\nu - \sum_{j=1}^{n_0-1} x_j\ \partial_{x_{j+1}}\ u_\nu(t,x) = \sum_{r=1}^{n} a_2^{\nu r} \partial^2_{x_1^2}(t,x)\ u_r(t,x),\ \ \nu = \overline{1,n.} \tag{3.1}$$

$$u_r(t,x)|_{t=\tau} = u_{0\,r}(x),\ x \epsilon R^{n_0},\ r = \overline{1,n},\ 0 \le \tau < t \le T, \tag{3.2}$$

where $u_{0r}(x)$ are sufficiently smooth compactly supported functions.

Let λ be roots $\lambda_1, ..., \lambda_n$ of equation $det\{(a_2^{\nu r}\ \ (is)^2)^n_{\nu,\,r=1} - \lambda I\} = 0$, where I is the identity matrix of order n, i is the imaginary unit and $Re\lambda(s) \le -\delta_0\ s_1^2,\ s_1 \in R^1$ with some constant $\delta_0 > 0$.

Using the Fourier transform we can reduce the Cauchy problem (3.1), (3.2) to the Cauchy problem for systems of differential equations in partial derivatives of the first order. For this components $u_1, ..., u_n$ solutions of Cauchy problem (3.1), (3.2) will be sought in the form of an inverse Fourier transform on s of unknown functions $v_1, ..., v_n$, namely

$$u(t,x) := F^{-1}[v_r(t,s)](t,x) := (2\pi)^{-n_0/2} \int\limits_{R^{n_0}} exp\{i(x,s)\} v_r(t,s) ds,$$

$$0 \leq \tau < t \leq T, x \in R^{n_0}, r = \overline{1,n}.$$

Taking into account the equality

$$\partial_t\, F^{-1}[v_r] = F^{-1}[\partial_t v_r]; x_j \partial_{x_{j+1}} F^{-1}[v_r] = F^{-1}[-s_{j+1}\partial_{s_j} v_r],$$
$$\partial^2_{x_1^2}\, F^{-1}[v_r] = F^{-1}[-s_1^2\, v_r], \partial_{x_1} F^{-1}[v_r] = F^{-1}[is_1 v_r]$$

we obtain for $v_1, ..., v_n$ the following Cauchy problem

$$\partial_t v_r(t,s) + \sum_{j=1}^{n_0-1} s_{j+1}\partial_{s_j}\, v_r\,(t,s) = -\sum_{k=1}^{n} a_2^{rk}\, s_1^2\, v_k\,(t,s). \tag{3.3}$$

$$v_r\,(t,s)\mid_{t=r} = v_{0r}(s),\ s \in R^{n_0},\ r = \overline{1,n},\ 0 \leq \tau < t \leq T. \tag{3.4}$$

Since functions $u_{0r}(x)$ are quite smooth and compactly supported their Fourier transforms are analytic functions for which the inequality true

$$|v_{0r}(s)| \leq c(1+|s|)^{-m},\ s \in R^{n_0},\ m \geq n_0 + 1, \tag{3.5}$$

where $v_{or}(s) := F[u_{0r}(x)]$.

In problems (3.3), (3.4) s^* - parameter. The system (3.3) consists of differential equations in partial derivatives of the first order and these equations have the same basic parts. According to [8, p. 146-148] this system is equivalent to a homogeneous linear differential equation with first-order partial derivatives for functions ω with $n + n_0$ independent variables $t, s_1, ..., s_{n_0-1}, v_1, ..., v_n$,

$$\partial_t \omega + \sum_{j=1}^{n_0-1} s_{j+1}\partial_{s_j}\omega + \sum_{r,\,l=1}^{n} a_2^{rl} s_1^2 v_r \partial_{v_l}\omega = 0,$$

which is equivalent to the system of ordinary differential equations:

$$dt = \frac{ds_1}{s_2} = \frac{ds_2}{s_3} = ... = \frac{ds_{n_0-1}}{s_{n_0}} = \frac{dv_1}{\sum\limits_{n=1}^{n} -a_2^{1r}\, s_1^2\, v_r} = ... = \frac{dv_n}{\sum\limits_{r=1}^{n} -a_2^{nr}\, s_1^2\, v_r}$$

.

Let us select $n_0 + n - 1$ independent integrals, in this system from $dt = \frac{ds_{n_0-1}}{s_{n_0}}$ we can find

$$s_{n_0-1} = ts_{n_0} + c_1 \tag{3.6}$$

and from $dt = \frac{ds_{n_0-2}}{s_{n_0-1}}$, taking into account (3.6), we have

$$s_{n_0-2} = t^2 s_{n_0}/2 + tc_1 + c_2 \tag{3.7}$$

and from $dt = \frac{ds_{n_0-k}}{s_{n_0-(k-1)}}$ for $k = \overline{3, n_0-1}$ we obtain

$$s_{n_0-k} = \frac{t^k}{k!} s_{n_0} + \frac{t^{k-1}}{(k-1)!} c_1 + \frac{t^{k-2}}{(k-2)!} c_2 + ... + c_k. \tag{3.8}$$

Using (3.6) - (3.8), we write

$$s = (s_1, s_2, ..., s_{n_0-(k-1)}, ..., s_{n_0}) = (\frac{t^{n_0-1}}{(n_0-1)!}s_{n_0} + \frac{t^{n_0-2}}{(n_0-2)!}c_1 + ... + c_{n_0-1}, ...,$$
$$\frac{t^{k-1}}{(k-1)!}s_{n_0} + \frac{t^{k-2}}{(k-2)!}c_1 + ... + c_{k-1}, ts_{n_0} + c_1, s_{n_0}). \quad (3.9)$$

Substituting (3.9) into the system of equations

$$dv_r = -\sum_{l=1}^{n} a_2^{rl} s_1^2 v_l dt, r = \overline{1,n} \quad (3.10)$$

we obtain the system of equations (3.10) for the characteristics of (3.6)-(3.8):

$$dv_r(t, P(t, s_{n_0}, c)) = -\sum_{l=1}^{n} a_2^{rl} \, (\frac{t^{n_0-1}}{(n_0-1)!}s_{n_0} + \sum_{k=2}^{n_0} \frac{t^{n_0-k}}{(n_0-k)!}c_{k-1})^2 v_l dt, \quad (3.11)$$

where

$$P(t, s_{n_0}, c) := (\frac{t^{n_0-1}}{(n_0-1)!}s_{n_0} + \sum_{k=2}^{n_0} \frac{t^{n_0-k}}{(n_0-k)!} \, c_{k-1}, ..., ts_{n_0} + c_1, s_{n_0}),$$

with the initial condition

$$v_r(t, P(t, s_{n_0}, c)|_{t=\tau} = v_{0r} \, (P(\tau, s_{n_0}, c)), \ r = \overline{1,n}. \quad (3.12)$$

Problem (3.11), (3.12) has a unique solution for $0 \leq \tau < t \leq T < +\infty$. Solution of Cauchy problem (3.16), (3.17) can be written as

$$v(t, P(t, s_{n_0}, c)) = Q(t, \tau, P(\tau, s_{n_0}, c))v_0(P(\tau, s_{n_0}, c)), \quad (3.13)$$

where $Q(t, \tau, P(\tau, s_{n_0}, c))$ is a normal matrix solutions of (3.11), $Q(t, \tau, P(\tau, s_{n_0}, c))|_{t=\tau} = I$.

Since the matrix

$$A(t) = (-a_2^{rl}(\frac{t^{n_0-1}}{(n_0-1)!}s_{n_0} + \sum_{k=2}^{n_0} \frac{t^{n_0-k}}{(n_0-k)!}c_{k-1})^2)_{r,\, l=1}^{n}$$

commutes with $\int\limits_{\tau}^{t} A(\tau)d\tau$, then

$$Q(t, \tau, P(\tau, s_{n_0}, c)) = exp\{-\int\limits_{\tau}^{t} A(\beta)d\beta\} = exp\{-A_1 \int\limits_{\tau}^{t} (\frac{\beta^{n_0-1}s_{n_0}}{(n_0-1)!} + \sum_{k=2}^{n_0} \frac{\beta^{n_{0-k}}}{(n_{0-k})!} \, c_{k-1})^2 d\beta\} \ ,$$

where $A_1 = (a_2^{r\ell})_{r,\, l=1}^{n}$.

We use the method of mathematical induction to find c_k, $k = \overline{1, n_0 - 1}$, with (3.6) - (3.7), for c_k is true c_k, $k = \overline{1, n_0 - 1}$, for c_k formula:

$$c_k = \sum_{j=0}^{k} (-t)^j s_{n_0-k+j}/j!, \ k = \overline{1, n_0}.$$

Valid, from (3.6) - (3.8) we have:

$$c_1 = s_{n_0-1} - ts_{no},$$

$$c_2 = s_{n_0} - tc_1 - \frac{t^2}{2!}\, s_{n_0} = s_{n_0-2} - ts_{n_0-1} + \frac{t^2}{2!} s_{n_0},$$

similar

$$c_3 = s_{n_0-3} - \frac{t^3}{3!} s_{n_0} - \frac{t^2}{2!} c_1 - tc_2 = s_{n_0-3} - ts_{n_0-2} + \frac{t^2}{2!} s_{n_0-1} - \frac{t^3}{3!} s_{n_0}, ...$$

Let $c_{k-1} = s_{n_0-k-1} - ts_{n_o-k+2} + ... + \frac{(-t)^j}{j!} s_{n_0-k+j+1} + ... + \frac{(-t)^{k-1}}{(k-1)!} s_{n_0}$, then for c_k we obtain

$$\begin{aligned} c_k &= s_{n_0-k} - \frac{t^k}{k!} s_{n_0} - \frac{t^{k-1}}{(k-1)!} c_1 - ... - tc_{k-1} = s_{n_0-k} - \frac{t^k}{(k)!} s_{n_0} - \frac{t^{k-1}}{(k-1)!} (s_{n_0-1} - ts_{n_0}) - t^{k-2} \\ &\times (s_{n_0-2} - ts_{n_0-1} + \frac{t^2}{2!} s_{n_0}) - ... - t(s_{n_0-k+1} - ts_{n_0-k+2} + ... + \frac{(-t)^j}{j!} s_{n_0-k+j+1} + ... \\ &+ \frac{(-t)^{k-1}}{(k-1)!} s_{n_0}) = s_{n_0-k} + s_{n_0} \frac{(-t)^k}{k!} + s_{n_0-1} \frac{(-t)^{k-1}}{(k-1)!} + ... + (-t) s_{n_0-k+1}, \end{aligned}$$

so we have

$$c_k = \sum_{j=0}^{k} (-t)^j s_{n_0-k+j}/j!,\ k = \overline{1,n}. \tag{3.14}$$

Substituting (3.14) into (3.13) we obtain

$$\begin{aligned} v(t,s) &= \exp\{-A_1 \int_{\tau}^{t} (\beta^{n_0-1} s_{n_0}/(n_0-1)! + \sum_{j=0}^{k} \beta^{n_0-k} (\sum_{j=0}^{k-1} s_{n_0-j} (-t)^{k-j-1}/(k-1-j)! \\ &\times ((n_0-1)!))^2 d\beta\}\ v_0(s_1 + (\tau-t)s_2 + (\tau-t)^2 s_3/2! + ... + (\tau-t)^{n_0-1} s_{n_0}/(n_0-1)!, s_2 \\ &+ (\tau-t)s_3 + ... + (\tau-t)^{n_0-2} s_{n_0}/(n_0-2)!, ..., s_{n_0-1} + (\tau-t)s_{n_0}, s_{n_0}). \end{aligned}$$

After the reduction of similar terms in the exponent exp, we will have

$$\begin{aligned} v(t,s) &= \exp\{-A_1 \int_{\tau}^{t} (s_1 + (\beta-t)s_2 + ... + (\beta-t)^{n_0-1} s_{n_0}/(n_0-1)!)^2 d\beta\} v_0(s_1 + \tau - t)s_2 + ... \\ &+ (\tau-t)^{n_0-1}! s_{n_0}/(n_0-1)!, s_2 + (\tau-t)s_3 + ... + (\tau-t)^{n_0-2} s_{n_0}/(n_0-2)!, ..., s_{n_0-1} \\ &+ (\tau-t)s_{n_0}, s_{n_0}). \end{aligned}$$

Find $u(t,x)$:

$$\begin{aligned} u(t,x) &= \frac{1}{(2\pi)^{n_0/2}} \int_{R^{n_0}} \exp\{i(x,s) - A_1 \int_{\tau}^{t} (s_1 + (\beta-t)s_2 + ... + (\beta-t)^{n_0-1} s_{n_0}/(n_0-1)!)^2 d\beta \\ &\times v_0(s_1 + (\tau-t)s_2 + ... + (\tau-t)^{n_0-1}\ s_{n_0}/(n_0-1)!, ...s_{n_0-1} + (\tau-t)s_{n_0}, s_{n_0}) ds. \end{aligned} \tag{3.15}$$

Changing the variables in (3.15) by

$$s_1 + (\tau-t)s_2 + ... + (\tau-t)^{n_0-1} s_{n_0}/(n_0-1)! = \alpha_1,$$
$$s_2 + (\tau-t)s_2 + ... + (\tau-t)^{n_0-2}\ s_{n_0}/(n_0-2)! = \alpha_2,$$

$$s_{n_0-1} + (\tau - t)s_{n_0} = \alpha_{n_0-1};$$

$$s_{n_0} = \alpha_{n_0},$$

or

$$s_1 = \alpha_1 - (\tau - t)\alpha_2 + ... + (-1)^{n-1} \frac{(\tau - t)^{n_0-1}}{(n_0-1)!}\alpha_{n_0},$$

$$s_2 = \alpha_2 - (\tau - t)\alpha_3 + ... + (-1)^{n-2} (\tau - t)^{n_0-2}\alpha_{n_0}/(n_0-2)!,$$

$$s_k = \alpha_k - (\tau - t)\alpha_{k+1} + ... + (-1)^{n_0-k} (\tau - t)^{n_0-k}/(n_0-k)!,$$

$$s_{n_0-1} = \alpha_{n_0-1} - (\tau - t)\alpha_{n_0},$$

$$s_{n_0} = \alpha_{n_0},$$

we obtain

$$\begin{aligned} u(t,x) &= \frac{1}{(2\pi)^{n_0/2}} \int\limits_{R^{n_0}} \exp\{i\alpha_1 x_1 + i\alpha_2(x_2 - (\tau - t)x_1) + i\alpha_3\,(x_3 - (\tau - t)x_2 + (\tau - t)^2 x_1/2!) \\ &+ ... + i\alpha_k \sum_{j=0}^{k-1} (-1)^j\, x_{k-j}(\tau - t)^j/j! + ... + i\alpha_{n_0} \sum_{j=0}^{n_0-1} (-1)^j\, x_{n_0-j}(\tau - t)^j/j! \\ &- A_1 \int\limits_{\tau}^{t} (\alpha_1 + (\beta - \tau)\alpha_2 + (\beta - \tau)^2\alpha_3/2! + ... + (\beta - \tau)^{n_0-1}\alpha_{n_0}/(n_0-1)!)^2 d\beta\} v_0(\alpha) d\alpha. \end{aligned}$$

$v_0(\alpha) = Fu_0(x)$, since

$$u(t,x) = \int_{R^{n_0}} G(t-\tau, x-\xi; x) u_0(\xi) d\xi, \tag{3.16}$$

where $G(t-\tau, x-\xi; x)$ is the fundamental solution of Cauchy problem and has the form:

$$\begin{aligned} G(t-\tau, x-\xi; x) &= (2\pi)^{-\frac{n_0}{2}} \int_{R^{n_0}} \exp\{i\alpha_1(x_1 - \xi_1) + i\alpha_2(x_2 - \xi_2 - (\tau - t)x_1) \\ &+ i\alpha_3(x_3 - \xi_3 - (\tau - t)x_2 + (\tau - t)^2 x_1/2!) + ... \\ &+ i\alpha_k(\sum_{j=0}^{k-1} (-1)^j\, x_{k-j}(\tau - t)^j/j! - \xi_k) + ... \\ &+ i\alpha_{n_0}(\sum_{j=0}^{n_0-1} (-1)^j\, x_{n_0-j}(\tau - t)^j/j! - \xi_{n_0}) - \int_{\tau}^{t} (\alpha_1 + (\beta - \tau)\alpha_2 + ... \\ &+ (\beta - \tau)^{n_0-1}\alpha_{n_0}/(n_0-1)!)^2 d\beta\} d\alpha. \end{aligned} \tag{3.17}$$

4. The Study of Behavior of the Fundamental Solution of Cauchy Problem

In order to investigate the behavior of $G(t-\tau, x-\xi; x)$ we compute integral

$$I = \int_{\tau}^{t} (\alpha_1 + (\beta-\tau)\alpha_2 + ... + (\beta-\tau)^{n_0-1}\alpha_{n_0}/(n_0-1)!)^2 d\beta. \tag{4.1}$$

Having replacement $(\beta-\tau)(\tau-t)^{-1} = \theta$, we obtain

$$I = \int_{0}^{1} (\alpha_1 + \theta(t-\tau)\alpha_2 + ... + \theta^{n_0-1}(t-\tau)^{n_0-1}\alpha_{n_0}/(n_0-1)!)^2 d\theta(t-\tau).$$

Denoting

$$\alpha_1(t-\tau)^{\frac{1}{2}} = s_1, \quad \alpha_2(t-\tau)^{\frac{3}{2}} = s_2, \quad ..., \quad \alpha_k(t-\tau)^{\frac{2k-1}{2}}/(k-1)! = s_k, \quad ...,$$

$$\alpha_n(t-\tau)^{\frac{2n_0-1}{2}}/(n_0-1)! = s_{n_0},$$

we have

$$\begin{aligned} I &= \int_0^1 (s_1 + \theta s_2 + ... + \theta^{n_0-1}s_{n_0})^2 d\theta = s_1^2 + s_2^2/3 + s_3^2/5 + ... + s_{n_0}^2/(2n_0-1) \\ &+ 2\sum_{j=2}^{n_0} s_1 s_j/j + 2\sum_{j=3}^{n_0} s_2 s_j/(j+1) + ... + 2\sum_{j=k+1}^{n_0} s_k s_j/(k+j-1) + ... \\ &+ 2s_{n_0-1}s_{n_0}/(2n_0-2). \end{aligned} \tag{4.2}$$

In (4.2) to select the perfect square $s_1, s_2, ..., s_{n_0}$, will have:

$$\begin{aligned} I &= (\sum_{j=1}^{n_0} s_j/j)^2 + 3(\sum_{j=2}^{n_0} \frac{(j-1)s_j}{j(j+1)})^2 + \frac{1}{180}(\sum_{k=3}^{n_0} \frac{s_k 30(k-1)(k-2)}{k(k+1)(k+2)})^2 \\ &+ (\sum_{k=4}^{n_0} \frac{s_k(k-1)(k-2)(k-3)}{k(k+1)(k+2)(k+3)})^2 + ... + (2j-1)(\sum_{k=j}^{n_0} \frac{s_k(k-1)...(k-(j-1))}{k(k+1)...(k+j-1)})^2 + ... \\ &+ (2n_0-3)(\sum_{k=-1}^{n_0} \frac{s_k(k-1)...(k-(n_0-2))}{k(k+1)...(k+n_0-1)})^2 + (2n_0-1)s_0^2 \frac{(n_0-1)^2(n_0-2)^2...2^2}{n_0^2(n_0+1)^2...(2n_0-1)^2} \end{aligned} \tag{4.3}$$

Using (4.3) $G(t-\tau, x-\xi; x)$ written as:

$$
\begin{aligned}
G(t-\tau, x-\xi; x) &= (2\pi)^{-n_0} \int\limits_{R^{n_0}} \exp\{is_1(x_1-\xi_1)(t-\tau)^{-1/2} + is_2(x_2-\xi_2-(\tau-t)x_1) \\
&\times (t-\tau)^{-3/2} + 2!is_3(x_3-\xi_3-(\tau-t)x_2+(t-\tau)^2x_1/2!)(t-\tau)^{-5/2} + \ldots \\
&+ (k-1)!is_k(\sum_{j=0}^{k-1}(-1)^j x_{k-j}(\tau-t)^j/j! - \xi_k)(t-\tau)^{-(2k-1)/2} \\
&+ is_{n_0}(\sum_{j=0}^{n_0-1}(-1)^j x_{n_0-1}(\tau-t)^j/j! - \xi_{n_0})(t-\tau)^{-(2n_0-1)/2}(n_0-1)! \\
&- A_1[(\sum_{j=1}^{n_0} s_j/j)^2 + 3(\sum_{j=2}^{n_0}\frac{(j-1)s_j}{j(j+1)})^2 + 5(\sum_{j=3}^{n_0}\frac{s_j(j-1)(j-2)}{j(j+1)(j+2)})^2 + \ldots \\
&+ (2k-1)(\sum_{j=k}^{n_0}\frac{s_j(j-1)\ldots(j-k+1)}{j(j+1)\ldots(j+k-1)})^2 + \ldots + (2n_0-3) \\
&\times (\sum_{j=n_0-1}^{n_0}\frac{s_j(j-1)\ldots(j-(n_0+2))}{j(j+1)\ldots(j+n_0-2)})^2 + (2n_0-1)(\frac{s_{n_0}(n_0-1)!}{n_0\ldots(2n_0-1)})^2\}]ds \\
&\times (t-\tau)^{-n_0^2/2}2!\ldots(n_0-1)! \qquad (4.4)
\end{aligned}
$$

Consider the system

$$
\begin{aligned}
\sum_{j=1}^{n_0}\frac{s_j}{j} &= \alpha_1, \\
\sum_{j=2}^{n_0}\frac{(j-1)s_j}{j(j+1)} &= \alpha_2, \\
&\ldots \\
\sum_{j=k}^{n_0}\frac{s_j(j-1)\ldots(j-k+1)}{j(j+1)\ldots(j+k-1)} &= \alpha_k, \\
&\ldots \\
\frac{s_{n_0-1}(n_0-2)!}{(n_0-1)\ldots(2n_0-3)} + \frac{s_{n_0}(n_0-1)!}{n_0\ldots(2n_0-2)} &= \alpha_{n_0-1} \\
\frac{s_{n_0}(n_0-1)!}{n_0\ldots(2n_0-1)} &= \alpha_{n_0}. \qquad (4.5)
\end{aligned}
$$

If we solve (4.5) we obtain

$$s_1 = \alpha_1 - 3\alpha_2 + 5\alpha_3 - 7\alpha_4 + ... + (-1)^{n_0-1}(2n_0 - 1)\alpha_{n_0},$$
$$\frac{s_2}{2\cdot 3} = \alpha_2 - 5\alpha_3 + \frac{4\cdot 7}{2!}\alpha_4 - \frac{4\cdot 5\cdot 9}{3!}\alpha_5 + \frac{4\cdot 5\cdot 6\cdot 11}{4!}\alpha_6 + ... + \frac{(-1)^{n_0-2}}{(n_0-2)!}4\cdot 5...n_0(2n_0-1)\alpha_{n_0},$$
$$\dots$$
$$\frac{s_k(k-1)!}{k...(2k-1)} = \alpha_k - (2k+1)\alpha_{k+1} + \frac{2k(2k+3)}{2!}\alpha_{k+2} - \frac{2k(2k+1)(2k+5)}{3!}\alpha_{k+3}$$
$$+ \frac{2k(2k+1)(2k+2)(2k+7)}{4!}\alpha_{k+4} + ... + \frac{(-1)^{j-k}2k(2k+1)...(j+k-2)(2j-1)}{(j-k)!}\alpha_j + ...$$
$$+ \frac{(-1)^{n_0-k}2k(2k+1)...(n_0+k-2)(2n_0-1)}{(n_0-k)!}\alpha_{n_0},$$
$$\dots$$
$$\frac{s_{n_0-1}(n_0-2)!}{(n_0-1)...(2n_0-3)} = \alpha_{n_0-1} - \alpha_{n_0}(2n_0-1),$$
$$\frac{s_{n_0}(n_0-1)!}{n_0...(2n_0-1)} = \alpha_{n_0}.$$

With this system we find $s_k(k-1)!, k = \overline{1, n_0}$ and substitute in (4.4).

$$\begin{aligned}
G(t-\tau, x-\xi; x) \;&=\; (2\pi)^{-n_0}\int\limits_{R^{n_0}} \exp\{-A_1\sum_{k=1}^{n_0}(2k-1)\alpha_k^2 + i[\alpha_1 - 3\alpha_2 + 5\alpha_3 + ...\\
&+\; (-1)^{n_0-1}(2n_0-1)\alpha_{n_0}](x_1-\xi_1)(t-\tau)^{-1/2} + 2\cdot 3(t-\tau)^{-3/2}(x_2-\xi_2\\
&-\; (\tau-t)x_1)i[\alpha_2 - 5\alpha_3 + \frac{4\cdot 7}{2!}\alpha_4 - \frac{4\cdot 5\cdot 9}{3!}\alpha_5 + ... + \frac{(-1)^{n_0-2}}{(n_0-2)!}4\cdot 5...\\
&\times\; n_0(2n_0-1)\alpha_{n_0}] + k...(2k-1)(t-\tau)^{-(2k-1)/2}i[\alpha_k - (2k+1)\alpha_{k+1}\\
&+\; \frac{2k(2k+3)}{2!}\alpha_{k+2} - \frac{2k(2k+1)(2k+5)}{3!}\alpha_{k+3}\frac{2k(2k+1)(2k+2)(2k+7)}{4!}\alpha_{k+4}\\
&+\; ... + \frac{(-1)^{j-k}2k(2k+1)...(j+k-2)(2j-1)}{(j-k)!}\alpha_j + ...\\
&+\; \frac{(-1)^{n_0-k}2k(2k+1)...(n_0+k-2)(2n_0-1)}{(n_0-k)!}\alpha_{n_0}]\\
&\times\; (\sum_{j=0}^{k_0-1}(-1)^j x_{k_0-j}(\tau-t)^j/j! - \xi_k) + ...\\
&+\; n_0(n_0+1)...(2n_0-1)(t-\tau)^{-(2n_0-1)/2}i\alpha_{n_0}\\
&\times\; (\sum_{j=0}^{n_0-1}(-1)^j x_{n_0-j}(\tau-t)^j/j! - \xi_{n_0})\}d\alpha(t-\tau)^{-n_0^2/2}\\
&\times\; \prod_{k=1}^{n_0}k(k+1)...(2k-1).
\end{aligned} \tag{4.6}$$

In (4.6) with respect to group the similar terms α_j, we have:

$$
\begin{aligned}
G(t-\tau,x &- \xi;x) = (2\pi)^{-n_0}\int\limits_{R^{n_0}} \exp\{-A_1\sum_{k=1}^{n_0}(2k-1)\alpha_k^2 + i\alpha_1(t-\tau)^{-1/2}(x_1-\xi_1)\\
&+ i\alpha_2(t-\tau)^{-3/2}6[x_2-\xi_2+(x_1+\xi_1)(t-\tau)/2] + i\alpha_3(t-\tau)^{-5/2}60[x_3-\xi_3\\
&+ (t-\tau)(x_2+\xi_2)/2+(t-\tau)^2(x_1-\xi_1)/12] + 1540i\alpha_4(t-\tau)^{-7/2}[x_4-\xi_4\\
&+ (x_3+\xi_3)(t-\tau)/2+(x_2-\xi_2)(t-\tau)^2/10+(t-\tau)^3(x_1+\xi_1)/120] + ...\\
&+ n_0(n_0+1)...(2n_0-1)i\alpha_{n_0}[\sum_{j=0}^{n_0-1} x_{n_0-j}(t-\tau)^j/j - \xi_{n_0} - (t-\tau)\\
&\times (\sum_{j=0}^{n_0-2} x_{n_0-j-1}(t-\tau)^j/j - \xi_{n_0-1}) + (t-\tau)^2\\
&\times (\sum_{j=0}^{n_0-3} x_{n_0-j-2}(t-\tau)^j/j - \xi_{n_0-2})(n-2)/4(2n_0-3) + ...\\
&+ (-1)^{(n_0-k)}\frac{(t-\tau)^{(n_0-k)}}{(n_0-k)!}\frac{2k...(2k+1)...(2k+(n_0-k)-2)(2k+2(n_0-k)-1)}{n_0...(2n_0-1)}\\
&\times (\sum_{j=0}^{k-1} x_{k-j}(t-\tau)^j/j! - \xi_k) + ... + (-1)^{(n_0-2)}\frac{(t-\tau)^{(n_0-2)}(x_2-\xi_2+(t-\tau)x_1)}{2(n+1)...(2n-3)}\\
&+ (-1)^{(n_0-1)}\frac{(t-\tau)^{(n_0-1)}(x_1-\xi_1)}{n_0...(2n_0-2)}]\}d\alpha(t-\tau)^{-n_0^2/2}\prod_{k=1}^{n_0} k(k+1)...(2k-1). \qquad (4.7)
\end{aligned}
$$

Remark 1. Each coefficient of the $i\alpha_k$ can be reduced to the form:

$$
\begin{aligned}
&i\alpha_k(t-\tau)^{-(2k-1)/2}k...(2k-1)[x_k-\xi_k+(t-\tau)(x_{k-1}+\xi_{k-1}/2)+...+(x_{k-j}-(-1)^j\xi_{k-j})\\
&\times (t-\tau)^j\,\frac{2j(2j+1)...(k+j-2)}{j!k...(2k-2)} + ... + (x_1-(-1)^{k-1}\xi_1)(t-\tau)^{k-1}\frac{1}{j!k(k+1)...(2k-2)}]\\
&= i\alpha_k k...(2k-1)[x_k-\xi_k+(t-\tau)(x_{k-1}-\xi_{k-1})/2+...+(x_{k-j}-(-1)^j\xi_{k-j})\\
&\times (t-\tau)\frac{(j+1)...(k+j-2)}{(j-1)!(k-1)k...(2k-3)} + \frac{(x_1-(-1)^{k-1}\xi_1)(t-\tau)^{k-1}}{2(k-1)k...(2k-3)}].
\end{aligned}
$$

From (4.7) it follows: $G(t-\tau,x-\xi;x)$ is the Fourier transform of

$$I_1(\sigma) = \exp\{-A_1\sum_{k=1}^{n_0}(2k-1)\alpha_k^2\},\ \sigma\in R^{n_0},$$

according to selected points, with parabolic [9], we obtain estimates for $I_1(\alpha+i\beta), \alpha\in R^{n_0}, \beta\in R^{n_0}$,

$$|I_1(\alpha + i\beta)| \le C \exp\{-c_0 \sum_{k=0}^{n_0} \alpha_k^2 + c \sum_{k=0}^{n_0} \beta_k^2\},$$

where positive constants C, c_0, c depends on n_0, n, constant parabolic δ_0, $\max\limits_{1 \le r,\, s \le n} |a_2^{rv}|$.

Fourier transform of I_1 is an entire function for which the derivatives satisfy estimation at $t > \tau, x \in R^{n_0}, \xi \in R^{n_0}$:

$$\begin{aligned}
\|\partial_{x_j}^m G(t &- \tau, x - \xi; x)| \le C_m (t-\tau)^{-\frac{n_0^2}{2} - \frac{(2j-1)m}{2}} \exp\{-c_0^* [|x_1 - \xi_1|^2 4^{-1} (t-\tau)^{-1} + 3|x_2 - \xi_2 \\
&+ (t-\tau)(x_1 + \xi_1)/2|^2 (t-\tau)^{-3} + 180|x_3 - \xi_3 + (x_2 + \xi_2)(t-\tau)/2 + (t-\tau)^2 (x_1 \\
&- \xi_1)/12|^2 (t-\tau)^{-5} + 25200|x_4 - \xi_4 + (x_3 + \xi_3)(t-\tau)/2 + (t-\tau)^2 (x_2 - \xi_2)/10 \\
&+ (x_1 + \xi_1)(t-\tau)^3/120|^2 (t-\tau)^{-7} + ... + (k-1)^2 k^2 ... (2k-3)^2 (2k-1)(t-\tau)^{-(2k-1)} \\
&\times |\sum_{j=0}^{k-1} x_{k-j}(t-\tau)^j/j! - \xi_k - (t-\tau)(\sum_{j=0}^{k-2} x_{k-1-j}(t-\tau)^j/j! - \xi_{k-1})/2 + ... \\
&+ (-1)^{k-l} \frac{(t-\tau)^{k-l}}{(k-l)!} \frac{2l(2l+1)...(2l+(k-l)-2)(2l+2(k-l)-1)}{k...(2k-1)} \\
&\times (\sum_{j=0}^{l-1} x_{l-j}(t-\tau)^j/j! - \xi_l) + ... + \frac{(-1)^{k-2}(t-\tau)^{k-2}(x_2 - \xi_2 + (t-\tau)x_1)}{2(k+1)...(2k-3)} \\
&+ \frac{(-1)^{k-1}(t-\tau)^{k-1}(x_1 - \xi_1)}{k...(2k-2)}|^2 + ... + (n_0-1)^2 n_0^2 (n_0+1)^2 ... (2n_0-3)^2 \\
&\times (2n_0 - 1)(t-\tau)^{-\frac{2n_0-1}{2}} | \sum_{j=0}^{n_0-1} x_{n_0-j}(t-\tau)^j/j! - \xi_{n_0} \\
&- (t-\tau)(\sum_{j=0}^{n_0-2} x_{n_0-1-j}(t-\tau)^j/j! - \xi_{n_0-1})/2 + (t-\tau)^2 \\
&\times (\sum_{j=0}^{n_0-3} x_{n_0-2-j}(t-\tau)^j/j! - \xi_{n_0-2}) \frac{(n_0-2)}{4(2n_0-3)} + ... + (-1)^{n_0-k} \\
&\times (\sum_{j=0}^{k-1} x_{k-j}(t-\tau)^j/j! - \xi_k) \frac{(t-\tau)^{n_0-k}}{(n_0-k)!} \frac{2k...(n_0+k-2)}{n_0...(2n_0-2)} \\
&\times (\sum_{j=0}^{k-1} x_{k-j}(t-\tau)^j/j! - \xi_k) + ... + (-1)^{n_0-2} \frac{(t-\tau)^{n_0-2}(x_2 - \xi_2 + (t-\tau)x_1)}{2(n_0+1)...(2n_0-3)} \\
&+ \frac{(-1)^{n_0-1}(t-\tau)^{n_0-1}(x_1 - \xi_1)}{n_0...(2n_0-3)}]^2 \qquad (4.8)
\end{aligned}$$

where positive constants C_m, c_0^* dependent $n_0, j, m, \delta, \sup\limits_{r,\, \nu} |a_2^{r\nu}|, T, j = \overline{1, n_0}$. After remark 1, formula (4.8) can be written as follows

$$\begin{aligned}
|\partial_{x_j}^m G(t &- \tau, x-\xi;x)| \leq C_m(t-\tau)^{-\frac{n_0^2+(2j-1)m}{2}} \exp\{-c_0^*[|x_1-\xi_1|^2 4^{-1}(t-\tau)^{-1} + 3|x_2-\xi_2 \\
&+ (x_1+\xi_1)2^{-1}(t-\tau)|^2(t-\tau)^{-3} + 180|x_3-\xi_3+(x_2+\xi_2)(t-\tau)/2+(t-\tau)^2 \\
&\times (x_1-\xi_1)/12|^2(t-\tau)^{-5} + 2520|x_4-\xi_4+(x_3+\xi_3)(t-\tau)/2 \\
&+ (t-\tau)^2(x_2-\xi_2)/10+(x_1+\xi_1)(t-\tau)^3/120|^2(t-\tau)^{-7}+... \\
&+ (k-1)^2...(2k-3)^2(2k-1)(t-\tau)^{-(2k-1)}|x_k-\xi_k+(t-\tau)(x_{k-1}+\xi_{k-1})/2+... \\
&+ (x_{k-j}-(-1)^j\xi_{k-j})(t-\tau)^j(j+1)...(k+j-2)/(j-1)!(k-1)k...(2k-3)+... \\
&+ (x_1-(-1)^{k-1}\xi_1)(t-\tau)^{k-1}/(2(k-1)k...(2k-3))|^2+... \\
&+ (n_0-1)^2...(2n_0-3)^2(2n_0-1)(t-\tau)^{-(2n_0-1)}|x_{n_0}-\xi_{n_0} \\
&+ (x_{n_0-1}+\xi_{n_0-1})(t-\tau)/2+...+(t-\tau)^{n_0-1}(x_1-(-1)^{n_0-1}\xi_1)/(2(n_0-1) \\
&\times (2n_0-3))|^2]\}, t-\tau>0, x\in R^{n_0}, \xi\in R^n, m\in N\cup\{0\}.
\end{aligned}$$

Remark 2. Estimates (4.8) are exact, because for system considered one equation $n=1$

$$\partial_t u(t,x) - \sum_{j=1}^{n_0-1} x_j \partial_{x_j+1} u(t,x) = \partial^2_{x_1^2} u(t,x)$$

we obtain $G(t-\tau, x-\xi; x)$ at $t>\tau$, in the form

$$\begin{aligned}
G(t &- \tau, x-\xi;x) = 2^{n_0}\pi^{-\frac{n_0}{2}} \prod_{k=1}^{n_0} k(k+1)...(2k-1)^{-\frac{1}{2}}(t-\tau)^{-\frac{2n_0-1}{2}} \exp\{-|x_1-\xi_1|^2 4^{-1} \\
&\times (t-\tau)^{-1} - 3|x_2-\xi_2+(x_1+\xi_1)2^{-1}(t-\tau)|^2(t-\tau)^{-3} - 180|x_3-\xi_3 \\
&+ (x_2+\xi_2)(t-\tau)2^{-1}+(t-\tau)^2(x_1-\xi_1)12^{-1}|^2(t-\tau)^{-5} - 2520|x_4-\xi_4 \\
&+ (x_3+\xi_3)(t-\tau)2^{-1}+(t-\tau)^2(x_2-\xi_2)10^{-1}+(x_1+\xi_1)(t-\tau)^3 120^{-1}|^2 \\
&\times (t-\tau)^{-7} - ... - k^2...(2k-3)^2(2k-1)(t-\tau)^{-(2k-1)}|x_k-\xi_k \\
&+ (t-\tau)(x_{k-1}+\xi_{k-1})2^{-1}+...+(x_{k-j}-(-1)^j\xi_{k-j})(t-\tau)^j(j+1)... \\
&\times (k+j-2)/(j-1)!(k-1)k...(2k-3)+...+(x_1-(-1)^{k-1}\xi_1)(t-\tau)^{k-1} \\
&\times (2(k-1)k...(2k-3))^{-1}|^2 - ... - n_0^2...(2n_0-2)^2(2n_0-1)(t-\tau)^{-(2n_0-1)} \\
&\times |x_{n_0}-\xi_{n_0}-(x_{n_0-1}+\xi_{n_0-1})(t-\tau)2^{-1}+... \\
&+ (t-\tau)^{n_0-1}(x_1-(-1)^{n_0-1}\xi_1)(2(n_0-1)...(2n_0-3))^{-1}|^2\} \qquad (4.9)
\end{aligned}$$

In particular, from (4.9) with $x=x_1,\ x_2=y,\ (n_0=2)$ have FSCP for the diffusion equation with inertia if $n_0=\overline{2,5}$ obtain the results of [10-12]. Repeating the arguments of this work for the equation

$$\partial_t u(t,x) - \sum_{\nu=1}^{p}\sum_{j=1}^{n_\nu-1} x_{\nu j}\partial_{x_{\nu(j+1)}} u(t,x) = \sum_{\nu=1}^{p} \partial^2_{x^2_{\nu 1}} u(t,x) + \sum_{\nu=p+1}^{m} \partial^2_{x^2_{\nu 1}} u(t,x), t>\tau,$$

$$x=(x_{11},...,x_{1n_1}; x_{21},...,x_{2n_2}; x_{p1},...,x_{pn_p}; x_{(p+1)1},...,x_{m1}),$$
$$n_1\geq n_2\geq ...\geq n_p>1, p>1, m\geq p,$$

we obtain an analogue of the formula (4.9):

$$
\begin{aligned}
G(t &- \tau, x-\xi; x) = \prod_{\nu=1}^{p}(2\sqrt{\pi})^{n_\nu+m-p}(t-\tau)^{-\frac{2n_\nu-1+m-p}{2}}\prod_{k=1}^{n_\nu}k(k+1)...(2k-1)^{-\frac{1}{2}}\\
&\times \exp\{-\sum_{j=1}^{p}[|x_{\nu 1}-\xi_{\nu 1}|^2 4^{-1}(t-\tau)^{-1}+3|x_{\nu 2}-\xi_{\nu 2}+(x_{\nu 1}+\xi_{\nu 1})2^{-1}(t-\tau)|^2(t-\tau)^{-3}\\
&+ 180|x_{\nu 3}-\xi_{\nu 3}+(x_{\nu 2}+\xi_{\nu 2})(t-\tau)2^{-1}+(t-\tau)^2(x_{\nu 1}-\xi_{\nu 1})12^{-1}|^2(t-\tau)^{-5}\\
&+ 2520|x_{\nu 4}-\xi_{\nu 4}+(x_{\nu 3}+\xi_{\nu 3})(t-\tau)2^{-1}+(t-\tau)^2(x_{\nu 2}-\xi_{\nu 2})10^{-1}+(x_{\nu 1}+\xi_{\nu 1})(t-\tau)^3\\
&\times 120^{-1}|^2(t-\tau)^{-7}+...+k^2...(2k-3)^2(2k-1)(t-\tau)^{-(2k-1)}|x_{\nu k}-\xi_{\nu k}+(t-\tau)(x_{\nu(k-1)}\\
&+ \xi_{\nu(k-1)})2^{-1}+...+(x_{\nu(k-j)}-(-1)^j\xi_{\nu(k-j)})(t-\tau)^j(j+1)...(k+j-2)/(j-1)!(k-1)k...\\
&\times (2k-3)+...+(x_{\nu 1}-(-1)^{k-1}\xi_{\nu 1})(t-\tau)^{k-1}(2(k-1)k...(2k-3))^{-1}|^2+...+n_\nu^2...\\
&\times (2n_\nu-2)^2(2n_\nu-1)(t-\tau)^{-(2n_\nu-1)}|x_{\nu n_\nu}-\xi_{\nu n_\nu}-(x_{\nu(n_\nu-1)}+\xi_{\nu(n_\nu-1)})(t-\tau)2^{-1}+...\\
&+ (t-\tau)^{n_\nu-1}(x_{\nu 1}-(-1)^{n_\nu-1}\xi_{\nu 1})(2(n_\nu-1)...(2n_\nu-3))^{-1}|^2]-\sum_{\nu=p-1}^{n}|x_{\nu 1}-\xi_{\nu 1}|^2 4^{-1}\\
&\times (t-\tau)^{-1}\}, x\in R^n,\ \xi\in R^n, n=\sum_{\nu=1}^{p}n_\nu+m-p.
\end{aligned}
$$

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Буртняк І.В., Малицька Г.П. Структура фундаментального розв'язку задачі Коші для систем Колмогорова другого порядку. *Журнал Прикарпатського університету імені Василя Стефаника,* **2** (4) (2015), 9-22.

Досліджується структура фундаментального розв'язку задачі Коші для одного класу систем ультрапараболічних рівнянь, що мають скінчену кількість груп змінних за якими вироджується параболічність.

Ключові слова: системи Колмогорова, фундаментальний розв'язок, вироджені параболічні рівняння.

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SYMMETRIC POLYNOMIALS AND HOLOMORPHIC FUNCTIONS ON INFINITE DIMENSIONAL SPACES

I.V. CHERNEGA

Abstract. A survey of general results about spectra of uniform algebras of symmetric holomorphic functions and algebras of symmetric analytic functions of bounded type on Banach spaces is given.

Keywords: polynomials and analytic functions on Banach spaces, symmetric polynomials, spectra of algebras.

1. SYMMETRIC POLYNOMIALS ON REARRANGEMENT-INVARIANT FUNCTION SPACES

Let X, Y be Banach spaces over the field $\mathbb{K}$ of real or complex numbers. A mapping $P : X \to Y$ is called an *n-homogeneous polynomial* if there exists a symmetric n-linear mapping $A : X^n \to Y$ such that for all $x \in X$ $P(x) = A(x, \ldots, x)$.

A *polynomial of degree n* on X is a finite sum of k-homogeneous polynomials, $k = 0, \ldots, n$. Let us denote by $\mathcal{P}(^n X, Y)$ the space of all n-homogeneous continuous polynomials $P : X \to Y$ and by $\mathcal{P}(X, Y)$ the space of all continuous polynomials.

It is well known ([13], XI §52) that for $n < \infty$ any symmetric polynomial on $\mathbb{C}^n$ is uniquely representable as a polynomial in the elementary symmetric polynomials $(G_i)_{i=1}^n$, $G_i(x) = \sum_{k_1 < \cdots < k_i} x_{k_1} \ldots x_{k_i}$.

Symmetric polynomials on ℓ_p and $L_p[0, 1]$ for $1 \le p < \infty$ were first studied by Nemirovski and Semenov in [16]. In [11] González, Gonzalo and Jaramillo investigated algebraic bases of various algebras of symmetric polynomials on so called rearrangement-invariant function spaces, that is spaces with some symmetric structure. Up to some inessential normalisation, the study of rearrangement-invariant function spaces is reduced to the study of the following three cases:

1. $I = \mathbb{N}$ and the mass of every point is one;
2. $I = [0, 1]$ with the usual Lebesgue measure;

3. $I = [0, \infty)$ with the usual Lebesgue measure.

We shall say that σ is an *automorphism* of I, if it ia a bijection of I, so that both σ and σ^{-1} are measurable and both preserve measure. We denote by $\mathcal{G}(I)$ the group of all automorphisms of I. If $X(I)$ is a rearrangement-invariant function space on I and $f \in X(I)$, then f is a real-valued measurable function on I and $f \circ \sigma \in X(I)$ for all $\sigma \in X(I)$. Also, there is an equivalent norm on $X(I)$ verifying that

$$\|f \circ \sigma\| = \|f\|$$

for all $\sigma \in \mathcal{G}(I)$ and all $f \in X(I)$. We always consider $X(I)$ endowed with this norm.

Following [16], we say that a polynomial P on $X(I)$ is *symmetric* if

$$P(f \circ \sigma) = P(f)$$

for all $\sigma \in \mathcal{G}(I)$ and all $f \in X(I)$.

In the same way, if $\mathcal{G}_0$ is a subgroup of $\mathcal{G}(I)$, a polynomial is said to be $\mathcal{G}_0$*-invariant* if $P(f) = P(f \circ \sigma)$ for all $\sigma \in \mathcal{G}_0$ and all $f \in X(I)$.

Let $X(I)$ be a rearrangement-invariant function space on I and consider the set

$$\mathcal{J}(X) = \{r \in \mathbb{N} : X(I) \subset L_r(I)\}.$$

Note that if $\mathcal{J}(X) \neq$ we can consider, for each $r \in \mathcal{J}(X)$, the polynomials

$$P_r(f) = \int_I f^r.$$

These are well-defined symmetric polynomials on $X(I)$ and we will call them the *elementary symmetric polynomials* on $X(I)$.

1.1. Symmetric Polynomials on Spaces with a Symmetric Basis

Let $X = X(\mathbb{N})$ be a Banach space with a symmetric basis $\{e_n\}$. A polynomial P on X is symmetric if for every permutation $\sigma \in \mathcal{G}(\mathbb{N})$

$$P\Big(\sum_{i=1}^{\infty} a_i e_i\Big) = P\Big(\sum_{i=1}^{\infty} a_i e_{\sigma(i)}\Big).$$

We consider the finite group $\mathcal{G}_n(\mathbb{N})$ of permutations of $\{1, \dots, n\}$ and the σ-finite group $\mathcal{G}_0(\mathbb{N}) = \cup_n \mathcal{G}_n(\mathbb{N})$ as subgroups of $\mathcal{G}(\mathbb{N})$. By continuity, a polynomial is symmetric if and only if it is $\mathcal{G}_0(\mathbb{N})$-invariant. Indeed, if P is $\mathcal{G}_0(\mathbb{N})$-invariant and $\sigma \in \mathcal{G}(\mathbb{N})$,

$$P\Big(\sum_{i=1}^{\infty} a_i e_i\Big) = \lim_{n\to\infty} P\Big(\sum_{i=1}^{n} a_i e_i\Big) = \lim_{n\to\infty} P\Big(\sum_{i=1}^{n} a_i e_{\sigma(i)}\Big) = P\Big(\sum_{i=1}^{\infty} a_i e_{\sigma(i)}\Big).$$

Recall that a sequence $\{x_n\}$ is said to have a *lower p-estimate* for some $p \geq 1$, if there is a constant $C > 0$ such that

$$C\Big(\sum_{i=1}^{n} |a_i|^p\Big)^{1/p} \leq \Big\|\sum_{i=1}^{n} a_i x_i\Big\|$$

for all $a_1, \dots, a_n \in \mathbb{R}$.

Note that $X \subset \ell_r$ if and only if the basis has a lower r-estimate, and therefore we have in this case

$$\mathcal{J}(X) = \{r \in \mathbb{N} : \{e_n\} \quad \text{has a lower r-estimate}\}.$$

Now we define

$$n_0(X) = \inf \mathcal{J}(X),$$

where we understand that the infimum of the empty set is ∞. The elementary symmetric polynomials are then

$$P_r\Big(\sum_{i=1}^{\infty} a_i e_i\Big) = \sum_{i=1}^{\infty} a_i^r,$$

where $r \geq n_0(X)$.

Theorem 1.1. [11] *Let X be a Banach space with a symmetric basis e_n, let P be a symmetric polynomial on X and consider $k = \deg P$ and $N = n_0(X)$.*

1. *If $k < N$, then $P = 0$.*
2. *If $k \geq N$, then there exists a real polynomial q of several real variables such that*

$$P\Big(\sum_{i=1}^{\infty} a_i e_i\Big) = q\Big(\sum_{i=1}^{\infty} a_i^N, \ldots, \sum_{i=1}^{\infty} a_i^k\Big)$$

for every $\sum_{i=1}^{\infty} a_i e_i \in X$.

1.2. Symmetric Polynomials on $X[0,1]$ and $X[0,\infty)$

Let $X[0,1]$ be a separable rearrangement-invariant function space on $[0,1]$. Note that the set $\mathcal{J}(X)$ is never empty since we always have $X[0,1] \subset L_1[0,1]$.

We define

$$n_\infty(X) = \sup\{r \in \mathbb{N} : X[0,1] \subset L_r[0,1]\}.$$

Therefore the elementary symmetric polynomials on $X[0,1]$ are

$$P_r(f) = \int_0^1 f^r$$

for each integer $r \leq n_\infty(X)$.

Theorem 1.2. [11] *Let $X[0,1]$ be a separable rearrangement-invariant function space on $[0,1]$ and consider the index $n_\infty(X)$ as above. Let P be a $\mathcal{G}_0[0,1]$-invariant polynomial on $X[0,1]$ and let $k = \deg P$. Then there exists a real polynomial q in several real variables such that*

$$P(f) = q\Big(\int_0^1 f, \ldots, \int_0^1 f^m\Big)$$

for all $f \in X$, where $m = \min\{n_\infty(X), k\}$.

Theorem 1.3. [11] *Let $X[0,\infty)$ be a separable rearrangement-invariant function space, let P be a $\mathcal{G}_0$-invariant polynomial on $X[0,\infty)$ and consider $k = \deg P$. Let n_0 and n_∞ be defined as above.*

1. *If either $n_0 > n_\infty$, or $k < n_0 \leq \infty$, then $P = 0$.*
2. *If $n_0 \leq n_\infty$ and $n_0 \leq k$, then there is a real polynomial q in several real variables such that*

$$P(f) = q\Big(\int_0^\infty f^{n_0}, \ldots, \int_0^\infty f^m\Big),$$

where $m = \min\{n_\infty, k\}$.

2. Uniform Algebras of Symmetric Holomorphic Functions

Let X be a Banach sequence space with a symmetric norm, that is, for all permutations $\sigma : \mathbb{N} \to \mathbb{N}$, and $x = (x_n) \in B$ also $(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, \ldots) \in B$, where B is an open unit ball.

A holomorphic function $f : B \to \mathbb{C}$ is called symmetric if for all $x \in B$ and all permutations $\sigma : \mathbb{N} \to \mathbb{N}$, the following holds:

$$f(x_1, \ldots, x_n, \ldots) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, \ldots).$$

Our interest throughout this section will be in the set

$$\mathcal{A}_{us}(B) = \{f : B \to \mathbb{C} | f \text{ is holomorphic, uniformly continuous, and symmetric on } B\}.$$

The following result is straightforward.

Proposition 2.1. [4] *$\mathcal{A}_{us}(B)$ is a unital commutative Banach algebra under the supremum norm. Each function $f \in \mathcal{A}_{us}(B)$ admits a unique (automatically symmetric) extension to $\overline{B}$.*

Let us give some examples of $\mathcal{A}_{us}(B)$ when B is the open unit ball of some classical Banach spaces X.

Example 2.2. $X = c_0$.

Theorem 2.3. [5] *Let $P : c_0 \to \mathbb{C}$ be an n-homogeneous polynomial and $\varepsilon > 0$. Then there is $N \in \mathbb{N}$ and an n-homogeneous polynomial $Q : \mathbb{C}^N \to \mathbb{C}$ such that for all $x = (x_1, \ldots, x_N, x_{N+1}, \ldots) \in B$, $|P(x) - Q(x_1, \ldots, x_N)| < \varepsilon$.*

Corollary 2.4. [4] *For all $n \in \mathbb{N}$, $n \geq 1$, the only n-homogeneous symmetric polynomial $P : c_0 \to \mathbb{C}$ is $P = 0$.*

Since any function $f \in \mathcal{A}_{us}(B)$ can be uniformly approximated on B by finite sums of symmetric homogeneous polynomials, it follows that $\mathcal{A}_{us}(B)$ consists of just the constant functions when B is the open unit ball of c_0.

Example 2.5. [4] $X = \ell_p$ for some p, $1 \leq p < \infty$.

The linear ($n = 1$) *case.* Let $\varphi \in \ell_p^*$ be a symmetric 1-homogeneous polynomial on ℓ_p; that is, φ is a symmetric continuous linear form. Since φ can be regarded as a point $(y_1, \ldots, y_m, \ldots) \in \ell_p^*$ and since $y_j = \varphi(e_1)$ for all j, we see that $y_1 = \ldots = y_m = \ldots$. Therefore, the set of symmetric linear forms φ on ℓ_1 consists of the 1-dimensional space $\{b(1, \ldots, 1, \ldots) | b \in \mathbb{C}\}$. For $p > 1$, the above shows that there are no non-trivial symmetric linear forms on ℓ_p.

The quadratic ($n = 2$) *case.* Let $P : \ell_p \to \mathbb{C}$ be a symmetric 2-homogeneous polynomial, and let $A : \ell_p \times \ell_p \to \mathbb{C}$ be the unique symmetric bilinear form associated to P, using the polarization formula and $P(x) = A(x, x)$ for all $x \in \ell_p$. Now, $P(e_1) = P(e_j)$ for all $j \in \mathbb{N}$. Moreover,

$$\begin{aligned} P(e_1 + e_2) &= A(e_1 + e_2, e_1 + e_2) = A(e_1, e_1) + 2A(e_1, e_2) + A(e_2, e_2) \\ &= P(e_1) + 2A(e_1, e_2) + P(e_2) \end{aligned}$$

and likewise

$$P(e_j + e_k) = P(e_j) + 2A(e_j, e_k) + P(e_k),$$

for all j and $k \in \mathbb{N}$. Therefore $A(e_j, e_k) = A(e_1, e_2)$.

So, for all N,

$$P(x_1, \ldots, x_N, 0, 0, \ldots) = a \sum_{j=1}^{N} x_j^2 + b \sum_{j \neq k} x_j x_k,$$

where $a = P(e_1)$ and $b = A(e_j, e_k)$.

From this, we can conclude, that for $X = \ell_1$, the space of symmetric 2-homogeneous polynomials on ℓ_1, $\mathcal{P}_s(^2\ell_1)$, is 2-dimensional with basis $\{\sum_j x_j^2, \sum_{j\neq k} x_j x_k\}$. On the other hand, the corresponding space $\mathcal{P}_s(^2\ell_2)$ of symmetric 2-homogeneous polynomials on ℓ_2, is 1-dimensional with basis $\{\sum_j x_j^2\}$. For $1 < p < 2$, $\mathcal{P}_s(^2\ell_p)$ is also the one-dimensional space generated by $\sum_j x_j^2$, while $\mathcal{P}_s(^2\ell_p) = \{0\}$ for $p > 2$.

This argument can be extended to all n and all p, and we can conclude that for all n, p, the space of symmetric n-homogeneous polynomials on ℓ_p, $\mathcal{P}_s(^n\ell_p)$, is finite dimensional. Consequently, since for all $f \in \mathcal{A}_{us}(B)$, f is a uniform limit of symmetric n-homogeneous polynomials, we have reasonably good knowledge about the functions in $\mathcal{A}_{us}(B)$. So we can say that $\mathcal{A}_{us}(B)$, for B the open unit ball of an ℓ_p space, is a "small" algebra.

2.1. THE SPECTRUM OF $\mathcal{A}_{us}(B)$

Recall that the *spectrum* (or *maximal ideal space*) of a Banach algebra $\mathcal{A}$ with identity e is the set $\mathcal{M}(\mathcal{A}) = \{\varphi : \mathcal{A} \to \mathbb{C} \quad | \varphi \quad \text{is a homomorphism and} \quad \varphi(e) = 1\}$. We recall that if $\varphi \in \mathcal{M}(\mathcal{A})$, then φ is automatically continuous with $\|\varphi\| = 1$. Moreover, when we consider it as a subset of $\mathcal{A}^*$ with the weak-star topology, $\mathcal{M}(\mathcal{A})$ is compact.

We will examine $\mathcal{M}(\mathcal{A}_{us}(B))$ when $B = B_{\ell_p}$. The most obvious element in $\mathcal{M}(\mathcal{A}_{us}(B))$ is the evaluation homomorphism δ_x at a point x of $\overline{B}$ (recalling that since the functions in $\mathcal{A}_{us}(B)$ are uniformly continuous, they have unique continuous extensions to $\overline{B}$). Of course, if $x, y \in B$ are such that y can be obtained from x by a permutation of its coordinates, then $\delta_x = \delta_y$. It is natural to wonder whether $\mathcal{M}(\mathcal{A}_{us}(B))$ consists of only the set of equivalence classes $\{\delta_{\tilde{x}} | x \in \overline{B}\}$, where $x \sim y$ means that x and y differ by a permutation.

Example 2.6. [1, 4]

For every $n \in \mathbb{N}$ define $F_n : B \to \mathbb{C}$ by $F_n(x) = \sum_{j=1}^{\infty} x_j^n$. To simplify, we take $B = B_{\ell_2}$ (so that F_n will be defined only for $n \geq 2$). It is known that the algebra generated by $\{F_n | n \geq 2\}$ is dense in $\mathcal{A}_{us}(B)$. For each $k \in \mathbb{N}$, let

$$v_k = \frac{1}{\sqrt{k}}(e_1 + ... + e_k).$$

It is routine that each v_k has norm 1, that $\delta_{v_k}(F_2) = 1$ for all $k \in \mathbb{N}$, and that for all $n \geq 3$,

$$\delta_{v_k}(F_n) = F_n(v_k) = \frac{1}{(\sqrt{k})^n} k \to 0 \text{ as } k \to \infty.$$

Since $\mathcal{M}(\mathcal{A}_{us}(B))$ is compact, the set $\{\delta_{v_k} | k \in \mathbb{N}\}$ has an accumulation point $\varphi \in \mathcal{M}(\mathcal{A}_{us}(B))$. It is clear that $\varphi(F_2) = 1$ and $\varphi(F_n) = 0$ for all $n \geq 3$. It is not difficult to verify that $\varphi \neq \delta_x$ for every $x \in \overline{B}$. This construction could be altered slightly, by letting $v_k = \frac{1}{\sqrt{k}}(\alpha_1 e_1 + ... + \alpha_k e_k)$, where each $|\alpha_j| \leq 1$. Thus, with this method we give a small number of additional homomorphisms in $\mathcal{M}(\mathcal{A}_{us}(B))$ that do not correspond to point evaluations.

It should be mentioned that it is not known whether $\mathcal{M}(\mathcal{A}_{us}(B_{\ell_p}))$ contains other points. However, in [1] was given a different characterization of $\mathcal{M}(\mathcal{A}_{us}(B_{\ell_p}))$. In order to do this, we first simplify our notation by considering only B_{ℓ_1}. For each $n \in \mathbb{N}$, define $\mathcal{F}^n : B_{\ell_1} \to \mathbb{C}^n$ as follows:

$$\mathcal{F}^n(x) = (F_1(x), \ldots, F_n(x)) = \Big(\sum_j x_j, \ldots, \sum_j x_j^n\Big).$$

Let $D_n = \mathcal{F}^n(B_{\ell_1})$, and let $[D_n]$ be the polynomially convex hull of D_n (see, e.g., [12]). Let

$$\Sigma_1 = \{(b_i)_{i=1}^\infty \in \ell_\infty : (b_i)_{i=1}^n \in [D_n], \text{ for all } n \in \mathbb{N}.\}$$

In other words, Σ_1 is the inverse limit of the sets $[D_n]$, endowed with the natural inverse limit topology.

Theorem 2.7. [1, 4] Σ_1 *is homeomorphic to* $\mathcal{M}(\mathcal{A}_{us}(B_{\ell_1}))$.

The analogous results, and the analogous definitions, are valid for Σ_p and $\mathcal{M}(\mathcal{A}_{us}(B_{\ell_p}))$.

The basic steps in the proof of Theorem 2.7 are as follows: First, since the algebra generated by $\{F_n | n \geq 1\}$ is dense in $\mathcal{A}_{us}(B_{\ell_1})$, each homomorphism $\varphi \in \mathcal{M}(\mathcal{A}_{us}(B_{\ell_1}))$ is determined by its behavior on $\{F_n\}$. Next, every symmetric polynomial P on ℓ_1 can be written as $P = Q \circ \mathcal{F}^n$ for some $n \in \mathbb{N}$ and some polynomial $Q : \mathbb{C}^n \to \mathbb{C}$. Finally, to each $(b_i) \in \Sigma_1$, one associates $\varphi = \varphi_{(b_i)} : \mathcal{A}_{us}(B_{\ell_1}) \to \mathbb{C}$ by $\varphi(P) = Q(b_1, \ldots, b_n)$. This turns out to be a well-defined homomorphism, and the mapping $(b_i) \in \Sigma_1 \rightsquigarrow \varphi_{(b_i)} \in \mathcal{M}(\mathcal{A}_{us}(B_{\ell_1}))$ is a homeomorphism.

2.2. THE SPECTRUM OF $\mathcal{A}_{us}(B)$ IN THE FINITE DIMENSIONAL CASE

Let us turn to $\mathcal{A}_{us}(B)$, where B is the open unit ball of $\mathbb{C}^n$, endowed with a symmetric norm. Because of finite dimensionality, $\mathcal{A}_{us}(B) = \mathcal{A}_s(B)$, where $\mathcal{A}_s(B)$ is the Banach algebra of symmetric holomorphic functions on B that are continuous on $\overline{B}$.

Unlike the infinite dimensional case, the following result holds.

Theorem 2.8. [1, 4] *Every homomorphism* $\varphi : \mathcal{A}_s(B) \to \mathbb{C}$ *is an evaluation at some point of* $\overline{B}$.

We describe below the main ideas in the proof of this result.

Proposition 2.9. [1, 4] *Let* $C \subset \mathbb{C}^n$ *be a compact set. Then* C *is symmetric and polynomially convex if and only if* C *is polynomially convex with respect to only the symmetric polynomials.*

In other words, C is symmetric and polynomially convex if and only if

$$C = \{z_0 \in \mathbb{C}^n : |P(z_0)| \leq \sup_{z \in C} |P(z)|, \text{ for all symmetric polynomials } P\}.$$

For $i \in \mathbb{N}$, let

$$R_i(x) = \sum_{1 \leq k_1 \leq \ldots \leq k_i \leq n} x_{k_1} \cdots x_{k_j}.$$

Proposition 2.10. [1, 4] *Let* B *be the open unit ball of a symmetric norm on* $\mathbb{C}^n$. *Then the algebra generated by the symmetric polynomials* $R_1, \ldots, R_n$ *is dense in* $\mathcal{A}_s(B)$.

Lemma 2.11. *(Nullstellensatz for symmetric polynomials)*[1, 4] *Let* $P_1, \ldots, P_m$ *be symmetric polynomials on* $\mathbb{C}^n$ *such that*

$$\ker P_1 \cap \cdots \cap \ker P_m = \emptyset.$$

Then there are symmetric polynomials $Q_1, \ldots, Q_m$ *on* $\mathbb{C}^n$ *such that*

$$\sum_{j=1}^m P_j Q_j \equiv 1.$$

To prove Theorem 2.8, let us consider the symmetric polynomials $P_1 = R_1 - \varphi(R_1), \ldots, P_m = R_m - \varphi(R_m)$. If $\ker P_1 \cap \cdots \cap \ker P_m = \emptyset$, then Lemma 2.11 implies that there are symmetric polynomials $Q_1, \ldots, Q_m$ on $\mathbb{C}^n$ such that $\sum_{j=1}^m P_j Q_j \equiv 1$. This is impossible, since $\varphi(P_j Q_j) = 0$. Therefore, there exists some $x \in \mathbb{C}^n$ such that $P_j(x) = 0$ for all j, which means $\varphi(R_j) = R_j(x)$ for all j. By Proposition 2.10, $\varphi(P) = P(x)$, for all symmetric polynomials $P : \mathbb{C}^n \to \mathbb{C}$.

So, for all such P, $|\varphi(P)| = |P(x)| \leq \|P\|$. This means that x belongs to the symmetrical polynomial convex hull of $\overline{B}$. Since $\overline{B}$ is symmetric and convex, it is symmetrically polynomially convex (by Proposition 2.9). Thus $x \in \overline{B}$. □

3. THE ALGEBRA OF SYMMETRIC ANALYTIC FUNCTIONS ON ℓ_p

Let us denote by $\mathcal{H}_{bs}(\ell_p)$ the algebra of all symmetric analytic functions on ℓ_p that are bounded on bounded sets endowed with the topology of the uniform convergence on bounded sets and by $\mathcal{M}_{bs}(\ell_p)$ the spectrum of $\mathcal{H}_{bs}(\ell_p)$, that is, the set of all non-zero continuous complex-valued homomorphisms.

3.1. THE RADIUS FUNCTION ON $\mathcal{M}_{bs}(\ell_p)$

Following [3] we define the *radius function* R on $\mathcal{M}_{bs}(\ell_p)$ by assigning to any complex homomorphism $\phi \in \mathcal{M}_{bs}(\ell_p)$ the infimum $R(\phi)$ of all r such that ϕ is continuous with respect to the norm of uniform convergence on the ball rB_{ℓ_p}, that is $|\phi(f)| \leq C_r \|f\|_r$. Further, we have $|\phi(f)| \leq \|f\|_{R(\phi)}$.

As in the non symmetric case, we obtain the following formula for the radius function

Proposition 3.1. [6] *Let $\phi \in \mathcal{M}_{bs}(\ell_p)$ then*

$$R(\phi) = \limsup_{n\to\infty} \|\phi_n\|^{1/n}, \tag{3.1}$$

where ϕ_n is the restriction of ϕ to $\mathcal{P}_s({}^n\ell_p)$ and $\|\phi_n\|$ is its corresponding norm.

Proof. To prove (3.1) we use arguments from [3, 2.3. Theorem]. Recall that

$$\|\phi_n\| = \sup\{|\phi_n(P)| : P \in \mathcal{P}_s({}^n\ell_p) \text{ with } \|P\| \leq 1\}.$$

Suppose that

$$0 < t < \limsup_{n\to\infty} \|\phi_n\|^{1/n}.$$

Then there is a sequence of homogeneous symmetric polynomials P_j of degree $n_j \to \infty$ such that $\|P_j\| = 1$ and $|\phi(P_j)| > t^{n_j}$. If $0 < r < t$, then by homogeneity,

$$\|P_j\|_r = \sup_{x \in rB_{\ell_p}} |P_j(x)| = r^{n_j},$$

so that

$$|\phi(P_j)| > (t/r)^{n_j} \|P_j\|_r,$$

and ϕ is not continuous for the $\| \, \|_r$ norm. It follows that $R(\phi) \geq r$, and on account of the arbitrary choice of r we obtain

$$R(\phi) \geq \limsup_{n\to\infty} \|\phi_n\|^{1/n}.$$

Let now be $s > \limsup_{n\to\infty} \|\phi_n\|^{1/n}$ so that $s^m \geq \|\phi_m\|$ for m large. Then there is $c \geq 1$ such that $\|\phi_m\| \leq cs^m$ for every m. If $r > s$ is arbitrary and $f \in \mathcal{H}_{bs}(\ell_p)$ has Taylor series expansion $f = \sum_{n=1}^{\infty} f_n$, then

$$r^m \|f_m\| = \|f_m\|_r \leq \|f\|_r, \quad m \geq 0.$$

Hence

$$|\phi(f_m)| \leq \|\phi_m\| \|f_m\| \leq \frac{cs^m}{r^m} \|f\|_r$$

and so

$$\|\phi(f)\| \leq c\Big(\sum \frac{s^m}{r^m}\Big) \|f\|_r.$$

Thus ϕ is continuous with respect to the uniform norm on rB, and $R(\phi) \leq r$. Since r and s are arbitrary,

$$R(\phi) \leq \limsup_{n\to\infty} \|\phi_n\|^{1/n}.$$

□

3.2. An Algebra of Symmetric Functions on the Polydisk of ℓ_1

Let us denote

$$\mathbb{D} = \Big\{x = \sum_{i=1}^{\infty} x_i e_i \in \ell_1 \colon \ \sup_i |x_i| < 1\Big\}.$$

It is easy to see that $\mathbb{D}$ is an open unbounded set. We shall call $\mathbb{D}$ the polydisk in ℓ_1.

Lemma 3.2. [6] *For every $x \in \mathbb{D}$ the sequence $\mathcal{F}(x) = (F_k(x))_{k=1}^{\infty}$ belongs to ℓ_1.*

Proof. Let us firstly consider $x \in \ell_1$, such that $\|x\| = \sum_{i=1}^{\infty} |x_i| < 1$ and let us calculate $\mathcal{F}(x) = (F_k(x))_{k=1}^{\infty}$. We have

$$\begin{aligned} \|\mathcal{F}(x)\| &= \sum_{k=1}^{\infty} |F_k(x)| = \sum_{k=1}^{\infty} \Big|\sum_{i=1}^{\infty} x_i^k\Big| \leq \sum_{k=1}^{\infty}\sum_{i=1}^{\infty} |x_i|^k \\ &\leq \sum_{k=1}^{\infty} \Big(\sum_{i=1}^{\infty} |x_i|\Big)^k = \sum_{k=1}^{\infty} \|x\|^k = \frac{\|x\|}{1-\|x\|} < \infty. \end{aligned}$$

In particular, $\|\mathcal{F}(\lambda e_k)\| = \frac{|\lambda|}{1-|\lambda|}$ for $|\lambda| < 1$.

If x is an arbitrary element in $\mathbb{D}$, pick $m \in \mathbb{N}$ so that $\sum_{i=m+1}^{\infty} |x_i| < 1$. Put $u = x - (x_1, \ldots, x_m, 0 \ldots)$ and and notice that $F_k(x) = F_k(x_1 e_1) + \cdots + F_k(x_m e_m) + F_k(u)$ with $\|x_k e_k\| < 1$, $k = 1, \ldots, m$ as well as $\|u\| < 1$. Also, $\|\mathcal{F}(x_k e_k)\| \leq \frac{\|x\|_\infty}{1-\|x\|_\infty}$. Hence,

$$\|\mathcal{F}(x)\| = \Big\|\sum_{k=1}^{m} \mathcal{F}(x_k e_k) + \mathcal{F}(u)\Big\| \leq \sum_{k=1}^{m} \|\mathcal{F}(x_k e_k)\| + \|\mathcal{F}(u)\| < \infty.$$

□

Note that $\mathcal{F}$ is an analytic mapping from $\mathbb{D}$ into ℓ_1 since $\mathcal{F}(x)$ can be represented as a convergent series $\mathcal{F}(x) = \sum_{k=1}^{\infty} F_k(x)e_k$ for every $x \in \mathbb{D}$ and $\mathcal{F}$ is bounded in a neighborhood of zero (see [9], p. 58).

Proposition 3.3. [6] *Let $g_1, g_2 \in \mathcal{H}_b(\ell_1)$. If $g_1 \neq g_2$, then there is $x \in \mathbb{D}$ such that $g_1(\mathcal{F}(x)) \neq g_2(\mathcal{F}(x))$.*

Proof. It is enough to show that if for some $g \in \mathcal{H}_b(\ell_1)$, we have $g(\mathcal{F}(x)) = 0$ $\forall x \in \mathbb{D}$, then $g(x) \equiv 0$.

Let $g(x) = \sum_{n=1}^{\infty} Q_n(x)$ where $Q_n \in \mathcal{P}(^n\ell_1)$ and

$$Q_n\Big(\sum_{n=1}^{\infty} x_i e_i\Big) = \sum_{k_1+\ldots+k_n=n} \sum_{i_1<\ldots<i_n} q_{n,i_1\ldots i_n} x_{i_1}^{k_1} \ldots x_{i_n}^{k_n}.$$

For any fixed $x \in \mathbb{D}$ and $t \in \mathbb{C}$ such that $tx \in \mathbb{D}$, let $g(\mathcal{F}(tx)) = \sum_{j=1}^{\infty} t^j r_j(x)$ be the Taylor series at the origin. Then

$$\sum_{n=1}^{\infty} Q_n(\mathcal{F}(tx)) = g(\mathcal{F}(tx)) = \sum_{j=1}^{\infty} t^j r_j(x).$$

Let us compute $r_m(x)$. We have

$$r_m(x) = \sum_{\substack{k<m \\ k_1i_1+\ldots k_ni_n=m}} q_{k,i_1\ldots i_n} F_{i_1}^{k_1}(x) \ldots F_{i_n}^{k_n}(x). \tag{3.2}$$

It is easy to see that the sum on the right hand side of (3.2) is finite.

Since $g(\mathcal{F}(x)) = 0$ for every $x \in \mathbb{D}$, then $r_m(x) = 0$ for every m. Further being $F_1, \ldots, F_n$ algebraically independent $q_{k,i_1\ldots i_n} = 0$ in (3.2) for an arbitrary $k < m$, $k_1i_1 + \ldots + k_ni_n = m$. As this is true for every m then $Q_n \equiv 0$ for $n \in \mathbb{N}$. So $g(x) \equiv 0$ on ℓ_1. ☐

Let us denote by $\mathcal{H}_s^{\ell_1}(\mathbb{D})$ the algebra of all symmetric analytic functions which can be represented by $f(x) = g(\mathcal{F}(x))$, where $g \in \mathcal{H}_b(\ell_1)$, $x \in \mathbb{D}$. In other words, $\mathcal{H}_s^{\ell_1}(\mathbb{D})$ is the range of the one-to-one composition operator $C_{\mathcal{F}}(g) = g \circ \mathcal{F}$ acting on $\mathcal{H}_b(\ell_1)$. According to Proposition 3.3 the correspondence $\Psi : f \mapsto g$ is a bijection from $\mathcal{H}_s^{\ell_1}(\mathbb{D})$ onto $\mathcal{H}_b(\ell_1)$. Thus we endow $\mathcal{H}_s^{\ell_1}(\mathbb{D})$ with the topology that turns the bijection Ψ an homeomorphism. This topology is the weakest topology on $\mathcal{H}_s^{\ell_1}(\mathbb{D})$ in which the following seminorms are continuous:

$$q_r(f) := \|(\Psi(f))\|_r = \|g\|_r = \sup_{\|x\|_{\ell_1} \leq r} |g(x)|, \quad r \in \mathbb{Q}.$$

Note that Ψ is a homomorphism of algebras. So we have proved the following proposition:

Proposition 3.4. [6] *There is an onto isometric homomorphism between the algebras $\mathcal{H}_s^{\ell_1}(\mathbb{D})$ and $\mathcal{H}_b(\ell_1)$.*

Corollary 3.5. [6] *The spectrum $\mathcal{M}(\mathcal{H}_s^{\ell_1}(\mathbb{D}))$ of $\mathcal{H}_s^{\ell_1}(\mathbb{D})$ can be identified with $\mathcal{M}_b(\ell_1)$. In particular, $\ell_1 \subset \mathcal{M}(\mathcal{H}_s^{\ell_1}(\mathbb{D}))$, that is, for arbitrary $z \in \ell_1$ there is a homomorphism $\psi_z \in \mathcal{M}(\mathcal{H}_s^{\ell_1}(\mathbb{D}))$, such that $\psi_z(f) = \Psi(f)(z)$.*

The following example shows that there exists a character on $\mathcal{H}_s^{\ell_1}(\mathbb{D})$, which is not an evaluation at any point of $\mathbb{D}$.

Example 3.6. [6] Let us consider a sequence of real numbers (a_n), $0 \le |a_n| < 1$ such that $(a_n) \in \ell_2 \setminus \ell_1$ and that the series $\sum_{n=1}^{\infty} a_n$ conditionally converges to some number C. Despite $(a_n) \notin \ell_1$, evaluations on (a_n) are determined for every symmetric polynomial on ℓ_1. In particular, $F_1((a_n)) = C$, $F_k((a_n)) = \sum a_n^k < \infty$ and $\{F_k((a_n))\}_{k=1}^{\infty} \in \ell_1$. So (a_n) "generates" a character on $\mathcal{H}_s^{\ell_1}(\mathbb{D})$ by the formula $\varphi(f) = \Psi(f)(\mathcal{F}((a_n)))$.

Since $(a_n) \in \ell_2$, then $F_k((a_{\pi(n)})) = F_k((a_n))$,$k > 1$. Notice that there exists a permutation on the set of positive integers, π, such that $\sum_{n=1}^{\infty} a_{\pi(n)} = C' \neq C$. For such a permutation π we may do the same construction as above and obtain a homomorphism φ_π "generated by evaluation at $(a_{\pi(n)})$", $\varphi(f) = \Psi(f)(\mathcal{F}((a_{\pi(n)})))$.

Let us suppose that there exist $x, y \in \mathbb{D}$ such that $\varphi(f) = f(x)$ and $\varphi_\pi(f) = f(y)$ for every function $f \in \mathcal{H}_s^{\ell_1}(\mathbb{D})$. Since $\varphi(F_k) = \varphi_\pi(F_k)$, $k \ge 2$, then by [1] Corollary 1.4, it follows that there is a permutation of the indices that transforms the sequence x into the sequence y. But this cannot be true, because $F_1(x) = \varphi(F_1) \neq \varphi_\pi(F_1) = F_1(y)$. Thus, at least one of the homomorphisms φ or φ_π is not an evaluation at some point of $\mathbb{D}$.

Note that the the homomorphism "generated by evaluation at (a_n)" is a character on $\mathcal{P}_s(\ell_1)$ too, but we do not know whether this character is continuous in the topology of uniform convergence on bounded sets.

3.3. The Symmetric Convolution

Recall that in [3] the convolution operation " $*$ " for elements φ, θ in the spectrum, $\mathcal{M}_b(X)$, of $\mathcal{H}_b(X)$, is defined by

$$(\varphi * \theta)(f) = \varphi(\theta(f(\cdot + x))), \text{ where } f \in \mathcal{H}_b(X). \tag{3.3}$$

In [6] we have introduced the analogous convolution in our symmetric setting.

It is easy to see that if f is a symmetric function on ℓ_p, then, in general, $f(\cdot + y)$ is not symmetric for a fixed y. However, it is possible to introduce an analogue of the translation operator which preserves the space of symmetric functions on ℓ_p.

Definition 3.7. [6] Let $x, y \in \ell_p$, $x = (x_1, x_2, \ldots,)$ and $y = (y_1, y_2, \ldots,)$. We define the *intertwining* of x and y, $x \bullet y \in \ell_p$, according to

$$x \bullet y = (x_1, y_1, x_2, y_2, \ldots,).$$

Let us indicate some elementary properties of the intertwining.

Proposition 3.8. [6] *Given $x, y \in \ell_p$ the following assertions hold.*

(1) *If $x = \sigma_1(u)$ and $y = \sigma_2(v)$, $\sigma_1, \sigma_2 \in \mathcal{G}$, then $x \bullet y = \sigma(u \bullet v)$ for some $\sigma \in \mathcal{G}$.*
(2) $\|x \bullet y\|^p = \|x\|^p + \|y\|^p$.
(3) *$F_n(x \bullet y) = F_n(x) + F_n(y)$ for every $n \ge p$.*

Proposition 3.9. [6] *If $f(x) \in \mathcal{H}_{bs}(\ell_p)$, then $f(x \bullet y) \in \mathcal{H}_{bs}(\ell_p)$ for every fixed $y \in \ell_p$.*

Proof. Note that $x \bullet y = x \bullet 0 + 0 \bullet y$ and that the map $x \mapsto x \bullet 0$ is linear. Thus the map $x \mapsto x \bullet y$ is analytic and maps bounded sets into bounded sets, and so is its composition with f. Moreover, $f(x \bullet y)$ is obviously symmetric. Hence it belongs to $\mathcal{H}_{bs}(\ell_p)$. $\square$

The mapping $f \mapsto T^s_y(f)$ where $T^s_y(f)(x) = f(x \bullet y)$ will be referred as to the intertwining operator. Observe that $T^s_x \circ T^s_y = T^s_{x \bullet y} = T^s_y \circ T^s_x$: Indeed, $[T^s_x \circ T^s_y](f)(z) = T^s_x[T^s_y(f)](z) = T^s_y(f)(z \bullet x) = f((z \bullet x) \bullet y)) = f(z \bullet (x \bullet y))$, since f is symmetric.

Proposition 3.10. [6] *For every $y \in \ell_p$, the intertwining operator T^s_y is a continuous endomorphism of $\mathcal{H}_{bs}(\ell_p)$.*

Proof. Evidently, T^s_y is linear and multiplicative. Let x belong to ℓ_p and $\|x\| \le r$. Then $\|x \bullet y\| \le \sqrt[p]{r^p + \|y\|^p}$ and

$$|T^s_y f(x)| \le \sup_{\|z\| \le \sqrt[p]{r^p + \|y\|^p}} |f(z)| = \|f\|_{\sqrt[p]{r^p + \|y\|^p}}. \tag{3.4}$$

So T^s_y is continuous. $\square$

Using the intertwining operator we can introduce a symmetric convolution on $\mathcal{H}_{bs}(\ell_p)'$. For any θ in $\mathcal{H}_{bs}(\ell_p)'$, according to (3.4) the radius function $R(\theta \circ T^s_y) \le \sqrt[p]{R(\theta)^p + \|y\|^p}$. Then arguing as in [3, 6.1. Theorem] , it turns out that for fixed $f \in \mathcal{H}_{bs}(\ell_p)$ the function $y \mapsto \theta \circ T^s_y(f)$ also belongs to $\mathcal{H}_{bs}(\ell_p)$.

Definition 3.11. For any ϕ and θ in $\mathcal{H}_{bs}(\ell_p)'$, their *symmetric convolution* is defined according to

$$(\phi \star \theta)(f) = \phi(y \mapsto \theta(T^s_y f)).$$

Corollary 3.12. [6] *If $\phi, \theta \in \mathcal{M}_{bs}(\ell_p)$, then $\phi \star \theta \in \mathcal{M}_{bs}(\ell_p)$.*

Proof. The multiplicativity of T^s_y yields that $\phi \star \theta$ is a character. Using inequality (3.4), we obtain that

$$R(\phi \star \theta) \le \sqrt[p]{R(\phi)^p + R(\theta)^p}.$$

Hence $\phi \star \theta \in \mathcal{M}_{bs}(\ell_p)$. $\square$

Theorem 3.13. [7] *a) For every $\varphi, \theta \in \mathcal{M}_{bs}(\ell_p)$ the following holds:*

$$(\varphi \star \theta)(F_k) = \varphi(F_k) + \theta(F_k). \tag{3.5}$$

b) The semigroup $(\mathcal{M}_{bs}(\ell_p), \star)$ is commutative, the evaluation at 0, δ_0, is its identity and the cancelation law holds.

Proof. Observe that for each element F_k in the algebraic basis of polynomials, $\{F_k\}$, we have

$$(\theta \star F_k)(x) = \theta(T^s_x(F_k)) = \theta(F_k(x) + F_k) = F_k(x) + \theta(F_k).$$

Therefore,

$$(\varphi \star \theta)(F_k) = \varphi(F_k + \theta(F_k)) = \varphi(F_k) + \theta(F_k).$$

To check that the convolution is commutative, that is, $\phi \star \theta = \theta \star \phi$, it suffices to prove it for symmetric polynomials, hence for the basis $\{F_k\}$. Bearing in mind (3.5) and also by exchanging parameters $(\theta \star \varphi)(F_k) = \theta(F_k) + \varphi(F_k) = (\varphi \star \theta)(F_k)$ as we wanted.

It also follows from (3.5) that the cancelation rule is valid for this convolution: If $\varphi \star \theta = \psi \star \theta$, then $\varphi(F_k) + \theta(F_k) = \psi(F_k) + \theta(F_k)$, hence $\varphi(F_k) = \psi(F_k)$, and thus, $\varphi = \psi$. $\square$

Example 3.14. [7] *There exist nontrivial elements in the semigroup* $(\mathcal{M}_{bs}(\ell_p), \star)$ *that are invertible*:

In [1, Example 3.1] it was constructed a continuous homomorphism $\varphi = \Psi_1$ on the uniform algebra $A_{us}(B_{\ell_p})$ such that $\varphi(F_p) = 1$ and $\varphi(F_i) = 0$ for all $i > p$. In a similar way, given $\lambda \in \mathbb{C}$ we can construct a continuous homomorphism Ψ_λ on the uniform algebra $A_{us}(|\lambda| B_{\ell_p})$ such that $\Psi_\lambda(F_p) = \lambda$ and $\Psi_\lambda(F_i) = 0$ for all $i > p$: It suffices to consider for each $n \in \mathbb{N}$, the element $v_n = \left(\frac{\lambda}{n}\right)^{1/p} (e_1 + \cdots + e_n)$ for which $F_p(v_n) = \lambda$, and $\lim_n F_j(v_n) = 0$. Now, the sequence $\{\delta_{v_n}\}$ has an accumulation point Ψ_λ in the spectrum of $A_{us}(|\lambda| B_{\ell_p})$. We use the notation ψ_λ for the restriction of Ψ_λ to the subalgebra $\mathcal{H}_{bs}(\ell_p)$ of $A_{us}(|\lambda| B_{\ell_p})$. It turns out that $\psi_\lambda \star \psi_{-\lambda} = \delta_0$ since for all elements F_j in the algebraic basis, $(\psi_\lambda \star \psi_{-\lambda})(F_j) = \psi_\lambda(F_j) + \psi_\lambda(F_j) = 0 = \delta_0(F_j)$.

Therefore, we obtain a complex line of invertible elements $\{\psi_\lambda \colon \lambda \in \mathbb{C}\}$.

As in the non-symmetric case [3] Theorem 5.5, the following holds:

Proposition 3.15. [7] *Every* $\varphi \in \mathcal{M}_{bs}(\ell_p)$ *lies in a schlicht complex line through* δ_0.

Proof. For every $z \in \mathbb{C}$, consider the composition operator $L_z : \mathcal{H}_{bs}(\ell_p) \to \mathcal{H}_{bs}(\ell_p)$ defined according to $L_z(f)((x_n)) := f((zx_n))$, and then, the restriction L_z^* to $\mathcal{M}_{bs}(\ell_p)$ of its transpose map. Now put $\varphi^z := L_z^*(\varphi) = \varphi \circ L_z$. Observe that $\varphi^z(F_k) = \varphi \circ L_z(F_k) = \varphi((F_k(z\cdot))) = z^k \varphi(F_k)$. Also, $\varphi^0 = \delta_0$.

For each $f \in \mathcal{H}_{bs}(\ell_p)$ the self-map of $\mathbb{C}$ defined according to $z \rightsquigarrow \varphi^z(f)$ is entire by [3] Lemma 5.4.(i). Therefore, the mapping $z \in \mathbb{C} \rightsquigarrow \varphi^z \in \mathcal{M}_{bs}(\ell_p)$ is analytic.

Since $\varphi \neq \delta_0$, the set $\Sigma := \{k \in \mathbb{N} : \varphi(F_k) \neq 0\}$ is non-empty. Let m be the first element of Σ, so that $\varphi(F_m) \neq 0$. Then if $\varphi^z = \varphi^w$, one has $z^m \varphi(F_m) = w^m \varphi(F_m)$, hence $z^m = w^m$. Taking the principal branch of the m^{th} root, the map $\xi \rightsquigarrow \varphi^{\sqrt[m]{\xi}}$ is one-to-one. □

Recall that a linear operator $T : \mathcal{H}_{bs}(\ell_p) \to \mathcal{H}_{bs}(\ell_p)$ is said to be a *convolution operator* if there is $\theta \in \mathcal{M}_{bs}(\ell_p)$ such that $Tf = \theta \star f$. Let us denote $H_{conv}(\ell_p) := \{T \in L(\mathcal{H}_{bs}(\ell_p)) : T \text{ is a convolution operator}\}$.

Proposition 3.16. [7] *A continuous homomorphism* $T : \mathcal{H}_{bs}(\ell_p) \to \mathcal{H}_{bs}(\ell_p)$ *is a convolution operator if, and only if, it commutes with all intertwining operators* T_y^s, $y \in \ell_p$.

Proof.- Assume there is $\theta \in \mathcal{M}_{bs}(\ell_p)$ such that $Tf = \theta \star f$. Fix $y \in \ell_p$. Then $[T \circ T_y^s](f)(x) = [T(T_y^s(f))](x) = [\theta \star T_y^s(f)](x) = \theta[T_x^s(T_y^s(f)] = \theta[T_{x\bullet y}^s(f)]$. On the other hand, $[T_y^s \circ T](f)(x) = [T_y^s(Tf)](x) = Tf(x \bullet y) = (\theta \star f)(x \bullet y) = \theta[T_{x\bullet y}^s(f)]$.

Conversely, set $\theta = \delta_0 \circ T$. Clearly, $\theta \in \mathcal{M}_{bs}(\ell_p)$. Let us check that $Tf = \theta \star f$: Indeed, $(\theta \star f)(x) = \theta[T_x^s(f)] = [T(T_x^s(f))](0) = [T_x^s(T(f))](0) = Tf(0 \bullet x) = Tf(x)$. □

Consider the mapping Λ defined by $\Lambda(\theta)(f) = \theta \star f$, that is,

$$\begin{array}{rccc} \Lambda : & \mathcal{M}_{bs}(\ell_p) & \to & H_{conv}(\ell_p) \\ & \theta & \mapsto & f \rightsquigarrow \theta \star f \equiv \Lambda(\theta)(f) \end{array}.$$

It is, clearly, bijective. Moreover we obtain a representation of the convolution semigroup

Proposition 3.17. [7] *The mapping* Λ *is an isomorphism from* $(\mathcal{M}_{bs}(\ell_p), \star)$ *into* $(H_{conv}(\ell_p), \circ)$ *where* $\circ$ *denotes the usual composition operation.*

Proof.- First, notice that using the above proposition,

$$\begin{aligned}\Lambda(\varphi\star\theta)(f)(x) &= [(\varphi\star\theta)\star f](x) = (\varphi\star\theta)(T_x^s f) = \varphi(\theta\star T_x^s f)\\ &= \varphi[\Lambda(\theta)(T_x^s f)] = \varphi[(\Lambda(\theta)\circ T_x^s)(f)] = \varphi[(T_x^s\circ\Lambda(\theta))(f)].\end{aligned}$$

On the other hand,

$$[\Lambda(\varphi)\circ\Lambda(\theta)](f)(x) = \Lambda(\varphi)[\Lambda(\theta)(f)](x) = [\varphi\star\Lambda(\theta)(f)](x) = \varphi[T_x^s(\Lambda(\theta)(f))].$$

Thus the statement follows. □

As a consequence, the homomorphism θ is invertible in $(\mathcal{M}_{bs}(\ell_p),\star)$, if, and only if, the convolution operator $\Lambda(\theta)$ is an algebraic isomorphism. Observe also that for $\psi\in\mathcal{M}_{bs}(\ell_p)$, one has

$$\psi\circ\Lambda(\theta) = \psi\star\theta,$$

because $[\psi\circ\Lambda(\theta)](f) = \psi[\Lambda(\theta)(f)] = \psi(\theta\star f) = (\psi\star\theta)(f)$.

Next we address the question of solving the equation $\varphi=\psi\star\theta$ for given $\varphi,\theta\in\mathcal{M}_{bs}(\ell_p)$. We begin with a general lemma.

Lemma 3.18. [7] *Let A, B be Fréchet algebras and $T: A\to B$ an onto homomorphism. Then T maps (closed) maximal ideals onto (closed) maximal ideals.*

Proof. Since T is onto, it maps ideals in A onto ideals in B. Let $\mathcal{J}\subset A$ be a maximal ideal, we prove that $T(\mathcal{J})$ is a maximal ideal in B : If $\mathcal{I}$ is another ideal with $T(\mathcal{J})\subset\mathcal{I}\subset B$, it turns out that for the ideal $T^{-1}(\mathcal{I})$, $\mathcal{J}\subset T^{-1}(T(\mathcal{J}))\subset T^{-1}(\mathcal{I})$, hence either $\mathcal{J}=T^{-1}(\mathcal{I})$, or $A=T^{-1}(\mathcal{I})$. That is, either $T(\mathcal{J})=\mathcal{I}$, or $B=\mathcal{I}$.

Let now $\varphi\in\mathcal{M}(A)$ and $\mathcal{J}=Ker(\varphi)$, a closed maximal ideal. Then $T(\mathcal{J})$ is a maximal ideal in B, so there is a character ψ on B such that $Ker(\psi)=T(\mathcal{J})$. Then $Ker(\varphi)\subset Ker(\psi\circ T)$, because if $\varphi(a)=0$, that is, $a\in\mathcal{J}$, we have $T(a)\in Ker(\psi)$. By the maximality, either $\varphi=\psi\circ T$, or $\psi\circ T=0$, hence $\psi=0$. In the former case, ψ is also continuous since being T an open mapping, if (b_n) is a null sequence in B, there is a null sequence $(a_n)\subset A$ such that $T(a_n)=b_n$; thus $\lim_n\psi(b_n)=\lim_n\psi\circ T(a_n)=\lim_n\varphi(a_n)=0$. □

Remark 3.19. *Let A, B be Fréchet algebras and $T: A\to B$ an onto homomorphism. If $T(Ker(\varphi))$ is a proper ideal, then there is a unique $\psi\in\mathcal{M}(B)$ such that $\varphi=\psi\circ T$.*

Corollary 3.20. [7] *Let $\theta\in\mathcal{M}_{bs}(\ell_p)$. Assume that $\Lambda(\theta)$ is onto. If $\Lambda(\theta)(Ker\varphi)$ is a proper ideal, then the equation $\varphi=\psi\star\theta$ has a unique solution. In case $\Lambda(\theta)(Ker\varphi)=\mathcal{H}_{bs}(\ell_p)$, then the equation $\varphi=\psi\star\theta$ has no solution.*

Proof. The first statement is just an application of the remark, since $\psi\star\theta=\psi\circ\Lambda(\theta)=\varphi$. For the second statement, if some solution ψ exists, then again $\psi\circ\Lambda(\theta)=\psi\star\theta=\varphi$, so $\psi(\mathcal{H}_{bs}(\ell_p)) = (\psi\circ\Lambda(\theta))((Ker\varphi)) = \varphi(Ker\varphi)=0$. Therefore, then also $\varphi=0$.

□

3.4. A WEAK POLYNOMIAL TOPOLOGY ON $\mathcal{M}_{bs}(\ell_p)$ [7]

Let us denote by w_p the topology in $\mathcal{M}_{bs}(\ell_p)$ generated by the following neighborhood basis:

$$U_{\varepsilon,k_1,\ldots,k_n}(\psi) = \{\psi\star\varphi\colon\ \varphi\in\mathcal{M}_{bs}(\ell_p)\quad |\varphi(F_{k_j})|<\varepsilon,\quad j=1,\ldots,n\}.$$

It is easy to check that the convolution operation is continuous for the w_p topology, since thanks to (3.5),

$$U_{\varepsilon/2,k_1,\ldots,k_n}(\theta)\star U_{\varepsilon/2,k_1,\ldots,k_n}(\psi)\subset U_{\varepsilon,k_1,\ldots,k_n}(\theta\star\psi).$$

We say that a function $f \in \mathcal{H}_{bs}(\ell_p)$ is *finitely generated* if there are a finite number of the basis functions $\{F_k\}$ and an entire function q such that $f = q(F_1, \ldots, F_j)$.

Theorem 3.21. *A function $f \in \mathcal{H}_{bs}(\ell_p)$ is w_p-continuous if and only if it is finitely generated.*

Proof. Clearly, every finitely generated function is w_p-continuous. Let us denote by V_n the finite dimensional subspace in ℓ_p spanned by the basis vectors $\{e_1, \ldots, e_n\}$. First we observe that if there is a positive integer m such that the restriction $f_{|V_n}$ of f to V_n is generated by the restrictions of $F_1, \ldots, F_m$ to V_n for every $n \geq m$, then f is finitely generated. Indeed, for given $n \geq k \geq m$ we can write

$$f_{|V_k}(x) = q_1(F_1(x), \ldots, F_m(x)) \quad \text{and} \quad f_{|V_n}(x) = q_2(F_1(x), \ldots, F_m(x))$$

for some entire functions q_1 and q_2 on $\mathbb{C}^n$. Since

$$\{(F_1(x), \ldots, F_m(x)) \colon x \in V_k\} = \mathbb{C}^m$$

(see e. g. [1]) and $f|_{V_n}$ is an extension of $f|_{V_k}$ we have $q_1(t_1, \ldots, t_n) = q_2(t_1, \ldots, t_n)$. Hence $f(x) = q_1(F_1(x), \ldots, F_m(x))$ on ℓ_p because $f(x)$ coincides with $q_1(F_1(x), \ldots, F_m(x))$ on the dense subset $\bigcup_n V_n$.

Let f be a w_p-continuous function in $\mathcal{H}_{bs}(\ell_p)$. Then f is bounded on a neighborhood $U_{\varepsilon,1,\ldots,m} = \{x \in \ell_p \colon |F_1(x)| < \varepsilon, \ldots, |F_m(x)| < \varepsilon\}$. For a given $n \geq m$ let

$$f|_{V_n}(x) = q(F_1(x), \ldots, F_m(x))$$

be the representation of $f|_{V_n}(x)$ for some entire function q on $\mathbb{C}^n$. Since $\{(F_1(x), \ldots, F_m(x)) \colon x \in V_n\} = \mathbb{C}^m$, $q(t_1, \ldots, t_n)$ must be bounded on the set $\{|t_1| < \varepsilon, \ldots, |t_m| < \varepsilon\}$. The Liouville Theorem implies $q(t_1, \ldots, t_n) = q(t_1, \ldots, t_m, 0 \ldots, 0)$, that is, $f|_{V_n}$ is generated by $F_1, \ldots, F_m$. Since it is true for every n, f is finitely generated. $\square$

For example $f(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{n!}$ is not w_p-continuous.

Proposition 3.22. *w_p is a Hausdorff topology.*

Proof. If $\varphi \neq \psi$, then there is a number k such that

$$|\varphi(F_k) - \psi(F_k)| = \rho > 0.$$

Let $\varepsilon = \rho/3$. Then for every θ_1 and θ_2 in $U_{\varepsilon,k}(0)$,

$$|(\varphi \star \theta_1)(F_k) - (\varphi \star \theta_2)(F_k)| = |(\varphi(F_k) - \psi(F_k)) - (\theta_2(F_k)) - \theta_1(F_k)| \geq \rho/3.$$

$\square$

Proposition 3.23. *On bounded sets of $\mathcal{M}_{bs}(\ell_p)$ the topology w_p is finer than the weak-star topology $w(\mathcal{M}_{bs}(\ell_p), \mathcal{H}_{bs}(\ell_p))$.*

Proof. Since $(\mathcal{M}_{bs}(\ell_p), w_p)$ is a first-countable space, it suffices to verify that for a bounded sequence $(\varphi_i)_i$ which is w_p convergent to some ψ, we have $\lim_i \varphi_i(f) = \psi(f)$ for each $f \in \mathcal{H}_{bs}(\ell_p)$: Indeed, by the Banach-Steinhaus theorem, it is enough to see that $\lim_i \varphi_i(P) = \psi(P)$ for each symmetric polynomial P. Being $\{F_k\}$ an algebraic basis for the symmetric polynomials, this will follow once we check that $\lim_i \varphi_i(F_k) = \psi(F_k)$ for each F_k. To see this, notice that given $\varepsilon > 0$, $\varphi_i \in U_{\varepsilon,k}$ for i large enough, that is, there is θ_i such that $\varphi_i = \psi \star \theta_i$ with $|\theta_i(F_k)| < \varepsilon$. Then, $|\varphi_i(F_k) - \psi(F_k)| = |\theta_i(F_K)| < \varepsilon$ for i large enough. $\square$

Proposition 3.24. *If $(\mathcal{M}_{bs}(\ell_p), \star)$ is a group, then w_p coincides with the weakest topology on $\mathcal{M}_{bs}(\ell_p)$ such that for every polynomial $P \in \mathcal{H}_{bs}(\ell_p)$ the Gelfand extension $\widehat{P}$ is continuous on $\mathcal{M}_{bs}(\ell_p)$.*

Proof. The sets $F_k^{-1}(B(F_k(\psi), \varepsilon))$ generate the weakest topology such that all $\widehat{P}$ are continuous. Let $\theta \in \mathcal{M}_{bs}(\ell_p)$ be such that $|F_k(\theta) - F_k(\psi)| < \varepsilon$. Set $\varphi = \theta \star \psi^{-1}$. Then $|F_k(\varphi)| = |F_k(\theta) - F_k(\psi)| < \varepsilon$ and $\theta = \psi \star \varphi$. □

3.5. Representations of the Convolution Semigroup $(\mathcal{M}_{bs}(\ell_1), \star)$[7]

In this subsection we consider the case $\mathcal{H}_{bs}(\ell_1)$. This algebra admits besides the power series basis $\{F_n\}$, another natural basis that is useful for us: It is given by the sequence $\{G_n\}$ defined by $G_0 = 1$, and

$$G_n(x) = \sum_{k_1 < \cdots < k_n}^{\infty} x_{k_1} \cdots x_{k_n},$$

and we refer to it as the *basis of elementary* symmetric polynomials.

Lemma 3.25. *We have that $\|G_n\| = 1/n!$*

Proof. To calculate the norm, it is enough to deal with vectors in the unit ball of ℓ_1 whose components are non-negative. And we may reduce ourselves to calculate it on L_m the linear span of $\{e_1, \ldots, e_m\}$ for $m \geq n$. We do the calculation in an inductive way over m.

Since $G_{n|_{L_m}}$ is homogeneous, its norm is achieved at points of norm 1. If $m = n$, then G_n is the product $x_1 \cdots x_n$. By using the Lagrange multipliers rule, we deduce that the maximum is attained at points with equal coordinates, that is at $\frac{1}{n}(e_1 + \cdots + e_n)$. Thus $|G_n(\frac{1}{n}, \overset{n}{\ldots}, \frac{1}{n}, 0, \ldots)| = 1/n^n \leq \frac{1}{n!}$.

Now for $m > n$, and $x \in L_m$, we have $G_n(x) = \sum_{k_1 < \cdots < k_n \leq m}^{\infty} x_{k_1} \cdots x_{k_n}$. Again the Lagrange multipliers rule leads to either some of the coordinates vanish or they are all equal, hence they have the same value $\frac{1}{m}$. In the first case, we are led back to some the previous inductive steps, with L_k with $k < m$, so the aimed inequality holds. While in the second one, we have

$$G_n(\frac{1}{m}, \overset{m}{\ldots}, \frac{1}{m}, 0, \ldots) = \begin{pmatrix} m \\ n \end{pmatrix} \frac{1}{m^n} \leq \frac{1}{n!}.$$

Moreover, $\|G_n\| \geq \lim_m \begin{pmatrix} m \\ n \end{pmatrix} \frac{1}{m^n} = \frac{1}{n!}$. This completes the proof. □

Let $\mathbb{C}\{t\}$ be the space of all power series over $\mathbb{C}$. We denote by $\mathcal{F}$ and $\mathcal{G}$ the following maps from $\mathcal{M}_{bs}(\ell_1)$ into $\mathbb{C}\{t\}$

$$\mathcal{F}(\varphi) = \sum_{n=1}^{\infty} t^{n-1} \varphi(F_n) \quad \text{and} \quad \mathcal{G}(\varphi) = \sum_{n=0}^{\infty} t^n \varphi(G_n).$$

Let us recall that every element $\varphi \in \mathcal{M}_{bs}(\ell_1)$ has a radius-function

$$R(\varphi) = \limsup_{n \to \infty} \|\varphi_n\|^{\frac{1}{n}} < \infty,$$

where φ_n is the restriction of φ to the subspace of n-homogeneous polynomials [6].

Proposition 3.26. *The mapping $\varphi \in \mathcal{M}_{bs}(\ell_1) \xrightarrow{\mathcal{G}} \mathcal{G}(\varphi) \in \mathcal{H}(\mathbb{C})$ is one-to-one and ranges into the subspace of entire functions on $\mathbb{C}$ of exponential type. The type of $\mathcal{G}(\varphi)$ is less than or equal to $R(\varphi)$.*

Proof. Using Lemma 3.25,

$$\begin{aligned}\limsup_{n\to\infty} \sqrt[n]{n!|\varphi_n(G_n)|} &\le \limsup_{n\to\infty} \sqrt[n]{n!\|\varphi_n\|\|G_n\|} \\ &= \limsup_{n\to\infty} \sqrt[n]{\|\varphi_n\|} = R(\varphi) < \infty,\end{aligned}$$

hence $\mathcal{G}(\varphi)$ is entire and of exponential type less than or equal to $R(\varphi)$. That $\mathcal{G}$ is one-to-one follows from the fact $\{G_n\}$ is a basis. $\square$

Theorem 3.27. *The following identities hold:*

(1) $\mathcal{F}(\varphi \star \theta) = \mathcal{F}(\varphi) + \mathcal{F}(\theta)$.
(2) $\mathcal{G}(\varphi \star \theta) = \mathcal{G}(\varphi)\mathcal{G}(\theta)$.

Proof. The first statement is a trivial conclusion of the properties of the convolution. To prove the second we observe that

$$G_n(x \bullet y) = \sum_{k=0}^{n} G_k(x)G_{n-k}(y).$$

Thus

$$(\theta \star G_n)(x) = \theta(T_x^s(G_n)) = \theta\Big(\sum_{k=0}^{n} G_k(x)G_{n-k}\Big) = \sum_{k=0}^{n} G_k(x)\theta(G_{n-k}).$$

Therefore,

$$(\varphi \star \theta)(G_n) = \varphi\Big(\sum_{k=0}^{n} G_k(x)\theta(G_{n-k})\Big) = \sum_{k=0}^{n} \varphi(G_k)\theta(G_{n-k}).$$

Hence, being the series absolutely convergent,

$$\begin{aligned}\mathcal{G}(\varphi)\mathcal{G}(\theta) &= \sum_{k=0}^{\infty} t^k\varphi(G_k) \sum_{m=0}^{\infty} t^m\theta(G_m) = \sum_{n=0}^{\infty} \sum_{k+m=n} t^n\varphi(G_k)\theta(G_m) \\ &= \sum_{n=0}^{\infty} t^n \sum_{k+m=n} \varphi(G_k)\theta(G_m) = \sum_{n=0}^{\infty} t^n(\varphi \star \theta)(G_n) = \mathcal{G}(\varphi \star \theta).\end{aligned}$$

$\square$

Example 3.28. Let ψ_λ be as defined in Example 3.14. We know that $\mathcal{F}(\psi_\lambda) = \lambda$. To find $\mathcal{G}(\psi_\lambda)$ note that

$$G_k(v_n) = \Big(\frac{\lambda}{n}\Big)^k \binom{n}{k}, \text{ hence } \varphi(G_k) = \lim_n G_k(v_n) = \frac{\lambda^k}{k!}$$

and so

$$\mathcal{G}(\psi_\lambda)(t) = \lim_{n\to\infty} \sum_{k=0}^{n} (\lambda t)^k \psi_\lambda(G_n) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(\lambda t)^k}{k!} = e^{\lambda t}.$$

According to well-known Newton's formula we can write for $x \in \ell_1$,

$$nG_n(x) = F_1(x)G_{n-1}(x) - F_2(x)G_{n-2}(x) + \cdots + (-1)^{n+1}F_n(x). \tag{3.6}$$

Moreover, if ξ is a complex homomorphism (not necessarily continuous) on the space of symmetric polynomials $\mathcal{P}_s(\ell_1)$, then

$$n\xi(G_n) = \xi(F_1)\xi(G_{n-1}) - \xi(F_2)\xi(G_{n-2}) + \cdots + (-1)^{n+1}\xi(F_n). \tag{3.7}$$

Next we point out the limitations of the construction's technique described in 3.14.

Remark 3.29. *Let ξ be a complex homomorphism on $\mathcal{P}_s(\ell_1)$ such that $\xi(F_m) = c \neq 0$ for some $m \geq 2$ and $\xi(F_n) = 0$ for $n \neq m$. Then ξ is not continuous.*

Proof. Using formula (3.7) we can see that

$$\xi(G_{km}) = (-1)^{m+1}\frac{\xi(F_m)\xi(G_{(k-1)m})}{km}$$

and $\xi(G_n) = 0$ if $n \neq km$ for some $k \in \mathbb{N}$. By induction we have

$$\xi(G_{km}) = \frac{\left((-1)^{m+1}c/m\right)^k}{k!}$$

and so

$$\mathcal{G}(\xi)(t) = 1 + \sum_{k=1}^{\infty} \frac{\left((-1)^{m+1}c/m\right)^k}{k!} t^{km} = 1 + \sum_{k=1}^{\infty} \frac{((-1)^{m+1}\frac{ct^m}{m})^k}{k!} = e^{\left((-1)^{m+1}\frac{ct^m}{m}\right)}.$$

Hence $\mathcal{G}(\xi)(t) = e^{-\frac{(-ct)^m}{m}} = e^{-\frac{(-c)^m}{m}t^m}$. Since $m \geq 2$, $\mathcal{G}(\xi)$ is not of exponential type. So if ξ were continuous, it could be extended to an element in $\mathcal{M}_{bs}(\ell_1)$, leading to a contradiction with Proposition 3.26. □

According to the Hadamard Factorization Theorem (see [14, p. 27]) the function of the exponential type $\mathcal{G}(\varphi)(t)$ is of the form

$$\mathcal{G}(\varphi)(t) = e^{\lambda t}\prod_{k=1}^{\infty}\Big(1 - \frac{t}{a_k}\Big)e^{t/a_k}, \tag{3.8}$$

where $\{a_k\}$ are the zeros of $\mathcal{G}(\varphi)(t)$. If $\sum |a_k|^{-1} < \infty$, then this representation can be reduced to

$$\mathcal{G}(\varphi)(t) = e^{\lambda t}\prod_{k=1}^{\infty}\Big(1 - \frac{t}{a_k}\Big). \tag{3.9}$$

Recall how ψ_λ was defined in Example 3.14.

Proposition 3.30. *If $\varphi \in (\mathcal{M}_{bs}(\ell_1), \star)$ is invertible, then $\varphi = \psi_\lambda$ for some λ. In particular, the semigroup $(\mathcal{M}_{bs}(\ell_1), \star)$ is not a group.*

Proof. If φ is invertible then $\mathcal{G}(\varphi)(t)$ is an invertible entire function of exponential type and so has no zeros. By Hadamard's factorization (3.8) we have that $\mathcal{G}(\varphi)(t) = e^{\lambda t}$ for some complex number λ. Hence $\varphi = \psi_\lambda$ by Proposition 3.26.

The evaluation $\delta_{(1,0\ldots,0,\ldots)}$ does not coincide with any ψ_λ since, for instance, $\psi_\lambda(F_2) = 0 \neq 1 = \delta_{(1,0\ldots,0,\ldots)}(F_2)$. □

Another consequence of our analysis is the following remark.

Corollary 3.31. *Let Φ be a homomorphism of $\mathcal{P}_s(\ell_1)$ to itself such that $\Phi(F_k) = -F_k$ for every k. Then Φ is discontinuous.*

Proof. If Φ is continuous it may be extended to continuous homomorphism $\widetilde{\Phi}$ of $\mathcal{H}_{bs}(\ell_1)$. Then for $x = (1, 0 \ldots, 0, \ldots)$, $\delta_x \star (\delta_x \circ \widetilde{\Phi}) = \delta_0$. However, this is impossible since δ_x is not invertible. □

We close this section by analyzing further the relationship established by the mapping $\mathcal{G}$.

It is known from Combinatorics (see e.g. [15, p. 3, 4]) that

$$\mathcal{G}(\delta_x)(t) = \prod_{k=1}^{\infty}(1 + x_k t) \quad \text{and} \quad \mathcal{F}(\delta_x)(t) = \sum_{k=1}^{\infty} \frac{x_k}{1 - x_k t} \tag{3.10}$$

for every $x \in c_{00}$. Formula (3.10) for $\mathcal{G}(\delta_x)$ is true for every $x \in \ell_1$: Indeed, for fixed t, both the infinite product and $\mathcal{G}(\delta_x)(t)$ are analytic functions on ℓ_1.

Taking into account formula (3.10) we can see that the zeros of $\mathcal{G}(\delta_x)(t)$ are $a_k = -1/x_k$ for $x_k \neq 0$. Conversely, if $f(t)$ is an entire function of exponential type which is equal to the right hand side of (3.9) with $\sum |a_k|^{-1} < \infty$, then for $\varphi \in \mathcal{M}_{bs}(\ell_1)$ given by $\varphi = \psi_\lambda \star \delta_x$, where $x \in \ell_1$, $x_k = -1/a_k$ and ψ_λ is defined in Example 3.14, it turns out that $\mathcal{G}(\varphi)(t) = f(t)$. So we have just to examine entire functions of exponential type with Hadamard canonical product

$$f(t) = \prod_{k=1}^{\infty} \left(1 - \frac{t}{a_k}\right) e^{t/a_k} \tag{3.11}$$

with $\sum |a_k|^{-1} = \infty$. Note first that the growth order of $f(t)$ is not greater than 1. According to Borel's theorem [14, p. 30] the series

$$\sum_{k=1}^{\infty} \frac{1}{|a_k|^{1+d}}$$

converges for every $d > 0$. Let

$$\Delta_f = \limsup_{n\to\infty} \frac{n}{|a_n|}, \qquad \eta_f = \limsup_{r\to\infty} \Big| \sum_{|a_n|<r} \frac{1}{a_n} \Big|$$

and $\gamma_f = \max(\Delta_f, \eta_f)$. Due to Lindelöf's theorem [14, p. 33] the type σ_f of f and γ_f simultaneously are equal either to zero, or to infinity, or to positive numbers. Hence $f(t)$ of the form (3.11) is a function of exponential type if and only if $\sum |a_k|^{-1-d}$ converges for every $d > 0$ and γ_f is finite.

Corollary 3.32. *If a sequence $(x_n) \notin \ell_p$ for some $p > 1$, then there is no $\varphi \in \mathcal{M}_{bs}(\ell_1)$ such that*

$$\varphi(F_k) = \sum_{n=1}^{\infty} x_n^k$$

for all k.

Let $x = (x_1, \ldots, x_n, \ldots)$ be a sequence of complex numbers such that $x \in \ell_{1+d}$ for every $d > 0$,

$$\limsup_{n\to\infty} n|x_n| < \infty, \qquad \limsup_{r\to 1} \Big| \sum_{\frac{1}{|x_n|}<r} x_n \Big| < \infty \tag{3.12}$$

and $\lambda \in \mathbb{C}$. Let us denote by $\delta_{(x,\lambda)}$ a homomorphism on the algebra of symmetric polynomials $\mathcal{P}_s(\ell_1)$ of the form

$$\delta_{(x,\lambda)}(F_1) = \lambda, \qquad \delta_{(x,\lambda)}(F_k) = \sum_{n=1}^{\infty} x_n^k, \quad k > 1.$$

Proposition 3.33. *Let $\varphi \in \mathcal{M}_{bs}(\ell_1)$. Then the restriction of φ to $\mathcal{P}_s(\ell_1)$ coincides with $\varphi_{(x,\lambda)}$ for some $\lambda \in \mathbb{C}$ and x satisfying (3.15).*

Proof. Consider the exponential type function $\mathcal{G}(\varphi)$ given by (3.8) and the corresponding sequence $x = (\frac{-1}{a_n})$.

If $x \in \ell_1$, then according to (3.9), $\varphi = \psi_\lambda \star \delta_x$. If $x \notin \ell_1$, then $\mathcal{G}(\varphi)(t) = e^{\lambda t}\prod_{n=1}^{\infty}\Big(1 + tx_n\Big)e^{-tx_n}$ and, on the other hand, $\mathcal{G}(\varphi)(t) = \sum_{n=0}^{\infty}\varphi(G_n)t^n$.

We have

$$\begin{aligned}\Big(e^{\lambda t}\prod_{n=1}^{\infty}\Big(1+tx_n\Big)e^{-tx_n}\Big)'_t &= \lambda e^{\lambda t}\prod_{n=1}^{\infty}\Big(1+tx_n\Big)e^{-tx_n} = e^{\lambda t}\Big(-tx_1^2e^{-tx_1}\prod_{n\neq 1}\Big(1+tx_n\Big)e^{-tx_n}\\ &- tx_2^2e^{-tx_2}\prod_{n\neq 2}\Big(1+tx_n\Big)e^{-tx_n} - \ldots\Big)\\ &= \lambda e^{\lambda t}\prod_{n=1}^{\infty}\Big(1+tx_n\Big)e^{-tx_n} - te^{\lambda t}\sum_{k=1}^{\infty}x_k^2e^{-tx_k}\prod_{n\neq k}\Big(1+tx_n\Big)e^{-tx_n}\end{aligned}$$

and

$$\Big(e^{\lambda t}\prod_{n=1}^{\infty}\Big(1+tx_n\Big)e^{-tx_n}\Big)'\Big|_{t=0} = \lambda.$$

So by the uniqueness of the Taylor coefficients, $\varphi(G_1) = \varphi(F_1) = \lambda$.

Now

$$\begin{aligned}\Big(e^{\lambda t}\prod_{n=1}^{\infty}\Big(1+tx_n\Big)e^{-tx_n}\Big)''_t &= \Big(\lambda e^{\lambda t}\prod_{n=1}^{\infty}\Big(1+tx_n\Big)e^{-tx_n}\Big)'_t\\ &- \Big(te^{\lambda t}\sum_{k=1}^{\infty}x_k^2e^{-tx_k}\prod_{n\neq k}\Big(1+tx_n\Big)e^{-tx_n}\Big)'_t\\ &= \lambda^2e^{\lambda t}\prod_{n=1}^{\infty}\Big(1+tx_n\Big)e^{-tx_n} - \lambda te^{\lambda t}\sum_{k=1}^{\infty}x_k^2e^{-tx_k}\prod_{n\neq k}\Big(1+tx_n\Big)e^{-tx_n}\\ &- e^{\lambda t}\sum_{k=1}^{\infty}x_k^2e^{-tx_k}\prod_{n\neq k}\Big(1+tx_n\Big)e^{-tx_n}\\ &- t\Big(e^{\lambda t}\sum_{k=1}^{\infty}x_k^2e^{-tx_k}\prod_{n\neq k}\Big(1+tx_n\Big)e^{-tx_n}\Big)'_t\end{aligned}$$

and

$$\Big(e^{\lambda t}\prod_{n=1}^{\infty}\Big(1+tx_n\Big)e^{-tx_n}\Big)''\Big|_{t=0} = \lambda^2 - \sum_{k=1}^{\infty}x_k^2.$$

Then

$$\varphi(G_2) = \frac{\lambda^2 - F_2(x)}{2} = \frac{(\varphi(F_1))^2 - F_2(x)}{2}.$$

On the other hand,

$$\varphi(G_2) = \frac{\varphi(F_1^2) - \varphi(F_2)}{2}$$

and we have

$$\varphi(F_2) = F_2(x).$$

Now using induction we obtain the required result. □

Question 3.34. Does the map $\mathcal{G}$ act onto the space of entire functions of exponential type?

3.6. THE MULTIPLICATIVE CONVOLUTION [8]

Definition 3.35. Let $x, y \in \ell_p$, $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots)$. We define the *multiplicative intertwining* of x and y, $x \diamond y$, as the resulting sequence of ordering the set $\{x_i y_j : i, j \in \mathbb{N}\}$ with one single index in some fixed order.

Note that for further consideration the order of numbering does not matter.

Proposition 3.36. *For arbitrary $x, y \in \ell_p$ we have*

(1) $x \diamond y \in \ell_p$ *and* $\|x \diamond y\| = \|x\|\|y\|$;
(2) $F_k(x \diamond y) = F_k(x)F_k(y)\ \forall k \geq \lceil p \rceil$.
(3) *If P is an n-homogeneous symmetric polynomial on ℓ_p and y is fixed, then the function $x \mapsto P(x \diamond y)$ is n-homogeneous.*

Proof. It is clear that $\|x \diamond y\|^p = \sum_{i,j} |x_i y_j|^p = \sum_i |x_i|^p \sum_i |y_j|^p = \|x\|^p \|y\|^p$. Also $F_k(x \diamond y) = \sum_{i,j}(x_i y_j)^k = \sum_i x_i^k \sum_j y_j^k = F_k(x)F_k(y)$. Statement (3) follows from the equality $\lambda(x \diamond y) = (\lambda x) \diamond y$. □

Given $y \in \ell_p$, the mapping $x \in \ell_p \overset{\pi_y}{\to} (x \diamond y) \in \ell_p$ is linear and continuous because of Proposition 3.36. Therefore if $f \in \mathcal{H}_{bs}(\ell_p)$, then $f \circ \pi_y \in \mathcal{H}_{bs}(\ell_p)$ because $f \circ \pi_y$ is analytic and bounded on bounded sets and clearly $f(\sigma(x) \diamond y) = f(x \diamond y)$ for every permutation $\sigma \in \mathcal{G}$. Thus if we denote $M_y(f) = f \circ \pi_y$, M_y is a composition operator on $\mathcal{H}_{bs}(\ell_p)$, that we will call the *multiplicative convolution operator*. Notice as well that $M_y = M_{\sigma(y)}$ for every permutation $\sigma \in \mathcal{G}$ and that $M_y(F_k) = F_k(y)F_k\ \ \forall k \geq \lceil p \rceil$.

Proposition 3.37. *For every $y \in \ell_p$ the multiplicative convolution operator M_y is a continuous homomorphism on $\mathcal{H}_{bs}(\ell_p)$.*

Note that in particular, if f_n is an n-homogeneous continuous polynomial, then $\|M_y(f_n)\| \leq \|f_n\|\|y\|^n$. And also that for $\lambda \in \mathbb{C}$, $M_{\lambda y}(f_n) = \lambda^n M_y(f_n)$, because $\pi_{\lambda y}(x) = \lambda\pi(x)$. Analogously, $M_{y+z}(f_n) = f_n \circ (\pi_y + \pi_z)$, because $\pi_{y+z} = \pi_y + \pi_z$. Therefore the mapping $y \in \ell_p \mapsto M_y(f_n)$ is an n-homogeneous continuous polynomial.

Recall that the *radius function* $R(\phi)$ of a complex homomorphism $\phi \in \mathcal{M}_{bs}(\ell_p)$ is the infimum of all r such that ϕ is continuous with respect to the norm of uniform convergence on the ball rB_{ℓ_p}, that is $|\phi(f)| \leq C_r\|f\|_r$. It is known that

$$R(\phi) = \limsup_{n\to\infty} \|\phi_n\|^{1/n},$$

where ϕ_n is the restriction of ϕ to $\mathcal{P}_s(^n\ell_p)$ and $\|\phi_n\|$ is its corresponding norm (see [6]).

Proposition 3.38. *For every $\theta \in \mathcal{H}_{bs}(\ell_p)'$ and every $y \in \ell_p$ the radius-function of the continuous homomorphism $\theta \circ M_y$ satisfies*

$$R(\theta \circ M_y) \leq R(\theta)\|y\|$$

and for fixed $f \in \mathcal{H}_{bs}(\ell_p)$ the function $y \mapsto \theta \circ M_y(f)$ also belongs to $\mathcal{H}_{bs}(\ell_p)$.

Proof. For a given $y \in \ell_p$, let $(\theta \circ M_y)_n$ (respectively, θ_n) be the restriction of $\theta \circ M_y$ (respectively, θ) to the subspace of n-homogeneous symmetric polynomials. Then we have

$$\|(\theta \circ M_y)_n\| = \sup_{\|f_n\| \leq 1} \left|\theta_n\Big(\frac{M_y(f_n)}{\|y\|^n}\Big)\right| \|y\|^n \leq \|\theta_n\| \|y\|^n.$$

So

$$R(\theta \circ M_y) \leq \limsup_{n \to \infty} (\|\theta_n\| \|y\|^n)^{1/n} = R(\theta)\|y\|.$$

Since the terms in the Taylor series of the function $y \mapsto \theta \circ M_y(f)$ are $y \mapsto \theta \circ M_y(f_n)$, where (f_n) are the terms in the Taylor series of f, the formula above proves the second statement. □

Using the multiplicative convolution operator we can introduce a multiplicative convolution on $\mathcal{H}_{bs}(\ell_p)'$.

Definition 3.39. Let $f \in \mathcal{H}_{bs}(\ell_p)$ and $\theta \in \mathcal{H}_{bs}(\ell_p)'$. The *multiplicative convolution* $\theta \Diamond f$ is defined as

$$(\theta \Diamond f)(x) = \theta[M_x(f)] \text{ for every } x \in \ell_p .$$

We have by Proposition 3.38, that $\theta \Diamond f \in \mathcal{H}_{bs}(\ell_p)$.

Definition 3.40. For arbitrary $\varphi, \theta \in \mathcal{H}_{bs}(\ell_p)'$ we define their *multiplicative convolution* $\varphi \Diamond \theta$ according to

$$(\varphi \Diamond \theta)(f) = \varphi(\theta \Diamond f) \text{ for every } f \in \mathcal{H}_{bs}(\ell_p).$$

For the evaluation homomorphism at y, δ_y, observe that

$$(\delta_y \Diamond f)(x) = \delta_y(M_x(f)) = (f \circ \pi_x)(y) = f(\pi_x(y)) = f(x \diamond y) = f(\pi_y(x)) = M_y(f)(x).$$

Hence, $\delta_x \Diamond \delta_y = \delta_{x \diamond y}$.

Proposition 3.41. *If $\varphi, \theta \in \mathcal{M}_{bs}(\ell_p)$, then $\varphi \Diamond \theta \in \mathcal{M}_{bs}(\ell_p)$.*

Proof. From the multiplicativity of M_y it follows that $\varphi \Diamond \theta$ is a character. Using arguments as in Proposition 3.38, we have that

$$R(\varphi \Diamond \theta) \leq R(\varphi) R(\theta).$$

Hence $\varphi \Diamond \theta \in \mathcal{M}_{bs}(\ell_p)$. □

Theorem 3.42.

$$1. \textit{If } \varphi, \theta \in \mathcal{M}_{bs}(\ell_p), \textit{ then } (\varphi \Diamond \theta)(F_k) = \varphi(F_k)\theta(F_k) \ \ \forall k \geq \lceil p \rceil. \tag{3.13}$$

2. *The semigroup $(\mathcal{M}_{bs}(\ell_p), \Diamond)$ is commutative and the evaluation at $x_0 = (1, 0, 0, \ldots)$, δ_{x_0}, is its identity.*

Proof. Let us take firstly $x, y \in \ell_p$ and $\delta_x, \delta_y \in \mathcal{M}_{bs}(\ell_p)$ the corresponding point evaluation homomorphisms. Then $(\delta_x \Diamond \delta_y)(F_k) = F_k(x \diamond y) = \sum x_i^k y_j^k = F_k(x) F_k(y)$.

Now let $\varphi, \theta \in \mathcal{M}_{bs}(\ell_p)$. Then

$$(\theta \Diamond F_k)(x) = \theta(M_x(F_k)) = \theta(F_k(x) F_k) = F_k(x)\theta(F_k).$$

So,

$$(\varphi \Diamond \theta)(F_k) = \varphi(F_k \theta(F_k)) = \varphi(F_k)\theta(F_k).$$

Exchanging parameters in (3.13) we get that

$$(\theta \Diamond \varphi)(F_k) = \theta(F_k)\varphi(F_k) = (\varphi \Diamond \theta)(F_k),$$

whence it follows that the multiplicative convolution is commutative for F_k. Since every symmetric polynomial is an algebraic combination of polynomials F_k and each function of $\mathcal{H}_{bs}(\ell_p)$ is uniformly approximated by symmetric polynomials, then the convolution operation is commutative. Analogously, $\Diamond$ is associative since

$$(\psi\Diamond(\varphi\Diamond\theta))\,(F_k) = \psi(F_k)\varphi(F_k)\theta(F_k) = ((\psi\Diamond\varphi)\Diamond\theta))(F_k).$$

Also from (3.13) it follows that the cancelation rule holds and δ_{x_0}, where $x_0 = (1,0,0,\ldots)$, is the identity. □

In [7] it was constructed a family $\{\psi_\lambda : \lambda \in \mathbb{C}\}$ of elements of the set $\mathcal{M}_{bs}(\ell_p)$ such that $\psi_\lambda(F_p) = \lambda$ and $\psi_\lambda(F_k) = 0$ for $k > p$. Let us recall the construction: Consider for each $n \in \mathbb{N}$, the element $v_n = \left(\frac{\lambda}{n}\right)^{1/p}(e_1 + \cdots + e_n)$ for which $F_p(v_n) = \lambda$, and $\lim_n F_j(v_n) = 0$ for $j > p$. Now, the sequence $\{\delta_{v_n}\}$ has an accumulation point ψ_λ in the spectrum for the pointwise convergence topology for which $\psi_\lambda(F_k) = 0$ for $k > p$ that prevents ψ_λ from being invertible because of (3.13).

Remark 3.43. The semigroup $(\mathcal{M}_{bs}(\ell_p), \Diamond)$ is not a group.

Recall that for any $\varphi, \theta \in \mathcal{M}_{bs}(\ell_p)$ and $f \in \mathcal{H}_{bs}(\ell_p)$, the symmetric convolution $\varphi \star \theta$ was defined in [6] as follows:

$$(\varphi \star \theta)(f) = \varphi(T_y^s(f)),$$

where $T_y^s(f)(x) = f(x \bullet y)$.

Proposition 3.44. *For arbitrary $\theta, \varphi, \psi \in \mathcal{M}_{bs}(\ell_p)$ the following equality holds:*

$$\theta\Diamond(\varphi \star \psi) = (\theta\Diamond\varphi) \star (\theta\Diamond\psi).$$

Proof. Indeed, using Theorem 3.42 and [7, Thm 1.5], we obtain that

$$\begin{aligned}((\theta\Diamond\varphi) \star (\theta\Diamond\psi))(F_k) &= (\theta\Diamond\varphi)(F_k) + (\theta\Diamond\psi)(F_k) = \theta(F_k)\varphi(F_k) + \theta(F_k)\psi(F_k) \\ &= \theta(F_k)(\varphi(F_k) + \psi(F_k)) = \theta(F_k)(\varphi \star \psi)(F_k) \\ &= \theta\Diamond(\varphi \star \psi)(F_k).\end{aligned}$$

□

Corollary 3.45. *The set $(\mathcal{M}_{bs}(\ell_p), \Diamond, \star)$ is a commutative semi-ring with identity.*

A linear operator $T : \mathcal{H}_{bs}(\ell_p) \to \mathcal{H}_{bs}(\ell_p)$ is called a *multiplicative convolution operator* if there exists $\theta \in \mathcal{M}_{bs}(\ell_p)$ such that $Tf = \theta\Diamond f$.

Proposition 3.46. *A continuous homomorphism $T : \mathcal{H}_{bs}(\ell_p) \to \mathcal{H}_{bs}(\ell_p)$ is a multiplicative convolution operator if and only if it commutes with all multiplicative operators M_y, $y \in \ell_p$.*

Proof. Suppose that there exists $\theta \in \mathcal{M}_{bs}(\ell_p)$ such that $Tf = \theta\Diamond f$. Fix $y \in \ell_p$. Then

$$[T \circ M_y](f)(x) = [T(M_y(f))](x) = [\theta\Diamond M_y(f)](x) = \theta[M_x(M_y(f)] = \theta[M_{x\diamond y}(f)].$$

On the other hand,

$$[M_y \circ T](f)(x) = [M_y(Tf)](x) = Tf(x \diamond y) = (\theta\Diamond f)(x \diamond y) = \theta[M_{x\diamond y}(f)].$$

Conversely, for $x_0 = (1,0,0,\ldots)$ we put $\theta = \delta_{x_0} \circ T$. Clearly, $\theta \in \mathcal{M}_{bs}(\ell_p)$. Let us check that $Tf = \theta\Diamond f$. Indeed, $(\theta\Diamond f)(x) = \theta[M_x(f)] = [T(M_x(f))](x_0) = [M_x(T(f))](x_0) = Tf(x_0 \diamond x) = Tf(x)$. □

Theorem 3.47. *A homomorphism $T : \mathcal{H}_{bs}(\ell_p) \to \mathcal{H}_{bs}(\ell_p)$ such that $T(F_k) = a_k F_k, \ \ k \geq \lceil p \rceil$, is continuous if and only if there exists $\varphi \in \mathcal{M}_{bs}(\ell_p)$ such that $\varphi(F_k) = a_k, \ \ k \geq \lceil p \rceil$.*

Proof. Let $\varphi \in \mathcal{M}_{bs}(\ell_p)$ with $\varphi(F_k) = a_k$. Then

$$(\varphi \Diamond F_k)(x) = \varphi(M_x(F_k)) = \varphi(F_k F_k(x)) = a_k F_k(x).$$

Thus if $Tf = \varphi \Diamond f$, T defines a continuous homomorphism and $T(F_k) = a_k F_k$.

Conversely, if such homomorphism T is continuous, then clearly T commutes with all M_y. By Proposition 3.46 it has the form $T(f) = \varphi \Diamond f$ for some $\varphi \in \mathcal{M}_{bs}(\ell_p)$. Thus, $T(F_k) = \varphi(F_k)F_k(x) = a_k F_k$, hence $\varphi(F_k) = a_k$. □

Proposition 3.48. *The identity is the only operator on $\mathcal{H}_{bs}(\ell_p)$ that is both a convolution and a multiplicative convolution operator.*

Proof. Let $T : \mathcal{H}_{bs}(\ell_p) \to \mathcal{H}_{bs}(\ell_p)$ be such an operator. Then there is $\theta \in \mathcal{M}_{bs}(\ell_p)$ such that $Tf = \theta \star f$ and T commutes with all M_y. In particular we have for all polynomials $F_k, \ \ k \geq \lceil p \rceil$, that

$$M_y(TF_k) = M_y(\theta \star F_k) = M_y(\theta(F_k) + F_k) = \theta(F_k) + M_y(F_k) = \theta(F_k) + F_k(y)F_k \quad \text{and}$$

$$T(M_y(F_k)) = T(F_k(y)F_k) = F_k(y)\theta \star F_k = F_k(y)(\theta(F_k) + F_k) \quad \text{coincide.}$$

Hence $\theta(F_k) = F_k(y)\theta(F_k)$, that leads to $\theta(F_k) = 0$, that in turn shows that $\theta = \delta_0$, or in other words, $T = Id$. □

3.7. The Case of ℓ_1 [8]

In this section we consider the algebra $\mathcal{H}_{bs}(\ell_1)$. In addition to the basis $\{F_n\}$, this algebra has a different natural basis that is given by the sequence $\{G_n\}$:

$$G_n(x) = \sum_{k_1 < \cdots < k_n}^{\infty} x_{k_1} \cdots x_{k_n}$$

and $G_0 := 1$.

According to [7] Lemma 3.1, $\|G_n\| = \frac{1}{n!}$, so it follows that for every $t \in \mathbb{C}$, the function $\sum_{n=0}^{\infty} t^n G_n \in \mathcal{H}_{bs}(\ell_1)$ and that such series converges uniformly on bounded subsets of ℓ_1. Thus if $\varphi \in \mathcal{M}_{bs}(\ell_1)$,

$$\mathcal{G}(\varphi)(t) = \varphi\Big(\sum_{n=0}^{\infty} t^n G_n\Big) = \sum_{n=0}^{\infty} t^n \varphi(G_n)$$

is well defined and as it was shown in [7, Proposition 3.2], the mapping

$$\varphi \in \mathcal{M}_{bs}(\ell_1) \stackrel{\mathcal{G}}{\to} \mathcal{G}(\varphi) \in H(\mathbb{C})$$

is one-to-one and ranges into the subspace of entire functions of exponential (finite) type. Whether $\mathcal{G}$ is an onto mapping was an open question there that we answer negatively here, see Corollary 3.52, using the multiplicative convolution we are dealing with.

Observe that for every $a \in \mathbb{C}$,

$$\Big(\delta_{(a,0,0,\ldots)} \Diamond \sum_{n=0}^{\infty} t^n G_n\Big)(x) = M_x\big(\sum_{n=0}^{\infty} t^n G_n\big)(a,0,0,\ldots) = \big(\sum_{n=0}^{\infty} t^n G_n\big)(x \diamond (a,0,0,\ldots))$$

$$= \sum_{n=0}^{\infty} t^n G_n(ax) = \sum_{n=0}^{\infty} t^n a^n G_n(x).$$

Therefore,

$$\mathcal{G}(\varphi\Diamond\delta_{(a,0,0,\ldots)})(t) = \varphi(\sum_{n=0}^{\infty} t^n a^n G_n) = \sum_{n=0}^{\infty} t^n a^n \varphi(G_n).$$

According to [7, Theorem 1.6 (a)] , $\delta_{(a,0,0,\ldots)} \star \delta_{(b,0,0,\ldots)} = \delta_{(a,b,0,0,\ldots)}$, consequently using Proposition 3.44 and [7, Theorem 3.3 (2)] ,

$$\mathcal{G}(\varphi\Diamond\delta_{(a,b,0,0,\ldots)})(t) = \mathcal{G}\Big((\varphi\Diamond\delta_{(a,0,0,\ldots)}) \star (\varphi\Diamond\delta_{(b,0,0,\ldots)})\Big)(t) = \mathcal{G}(\varphi\Diamond\delta_{(a,0,0,\ldots)})(t)\mathcal{G}(\varphi\Diamond\delta_{(b,0,0,\ldots)})(t) =$$

$$\sum_{n=0}^{\infty} t^n a^n \varphi(G_n) \cdot \sum_{n=0}^{\infty} t^n b^n \varphi(G_n).$$

Therefore,

$$\mathcal{G}(\varphi\Diamond\delta_{(x_1,x_2,\ldots,x_m,0,\ldots)})(t) = \prod_{k=1}^{m} \sum_{n=0}^{\infty} t^n x_k^n \varphi(G_n).$$

Further since the sequence $\left(\delta_{(x_1,x_2,\ldots,x_m,0,\ldots)}\right)_m$ is pointwise convergent to $\delta_{(x_1,x_2,\ldots,x_m\ldots)}$ in $M_{bs}(\ell_1)$ we have, bearing in mind the commutativity of $\Diamond$, that the sequence $\left(\varphi\Diamond\delta_{(x_1,x_2,\ldots,x_m,0,\ldots)}\right)_m$ is pointwise convergent to $\varphi\Diamond\delta_{(x_1,x_2,\ldots,x_m\ldots)}$. Thus

$$\mathcal{G}(\varphi\Diamond\delta_x)(t) = \prod_{k=1}^{\infty} \sum_{n=0}^{\infty} t^n x_k^n \varphi(G_n) \quad \text{for } x = (x_1, x_2, \ldots, x_m \ldots) \in \ell_1. \tag{3.14}$$

For the mentioned above family $\{\psi_\lambda : \lambda \in \mathbb{C}\}$, it was shown in [7] that $\mathcal{G}(\psi_\lambda)(t) = e^{\lambda t}$. Further, it is easy to see that

(1) $\psi_\lambda\Diamond\varphi(F_1) = \lambda\varphi(F_1)$.
(2) $\psi_\lambda\Diamond\varphi(F_k) = 0, \quad k > 1$.
(3) $\mathcal{G}(\psi_\lambda\Diamond\varphi) = e^{\lambda\varphi(F_1)t}$.

The following theorem might be of interest in Function Theory.

Theorem 3.49. *Let $g(t)$ and $h(t)$ be entire functions of exponential type of one complex variable such that $g(0) = h(0) = 1$. Let $\{a_n\}$ be zeros of $g(t)$ with $\sum_{n=1}^{\infty} \frac{1}{|a_n|} < \infty$ and let $\{b_n\}$ be zeros of $h(t)$ with $\sum_{n=1}^{\infty} \frac{1}{|b_n|} < \infty$. Then there exists a function of exponential type $u(t)$ with zeros $\{a_n b_m\}_{n,m}$, which can be represented as*

$$u(t) = \prod_{k=1}^{\infty} \sum_{n=0}^{\infty} \Big(-\frac{1}{a_k}\Big)^n h_n(t) = \prod_{k=1}^{\infty} \sum_{n=0}^{\infty} \Big(-\frac{1}{b_k}\Big)^n g_n(t).$$

Proof. By [7], $g(t) = \mathcal{G}(\delta_x)(t)$ and $h(t) = \mathcal{G}(\delta_y)(t)$, where $x, y \in \ell_1$, $x_n = -\frac{1}{a_n}$, $y_n = -\frac{1}{b_n}$. So $u(t) = \mathcal{G}(\delta_x\Diamond\delta_y)(t)$ and using (3.14) we obtain the statement of the theorem. ☐

Let $x = (x_1, \ldots, x_n, \ldots)$ be a sequence of complex numbers such that $x \in \ell_{1+d}$ for every $d > 0$,

$$\limsup_{n\to\infty} n|x_n| < \infty, \qquad \limsup_{r\to\infty} \Big|\sum_{\frac{1}{|x_n|}<r} x_n\Big| < \infty \tag{3.15}$$

(think for instance of $x_n = \frac{(-1)^n}{n}$) and $\lambda \in \mathbb{C}$. Let us denote by $\delta_{(x,\lambda)}$ a homomorphism on the algebra of symmetric polynomials $\mathcal{P}_s(\ell_1)$ of the form

$$\delta_{(x,\lambda)}(F_1) = \lambda, \qquad \delta_{(x,\lambda)}(F_k) = \sum_{n=1}^{\infty} x_n^k, \quad k > 1.$$

Recall that according to [14, p. 17], $\limsup_{n\to\infty} n|x_n|$ coincides with the so-called *upper density* of the sequence $(\frac{1}{x_n})$ that is defined by $\limsup_{r\to\infty} \frac{\mathbf{n}(r)}{r}$, where $\mathbf{n}(r)$ denotes the *counting number* of $(\frac{1}{x_n})$, that is, the number of terms of the sequence with absolute value not greater than r.

Proposition 3.50. [7, Proposition 3.9] *Let $\varphi \in \mathcal{M}_{bs}(\ell_1)$. Then the restriction of φ to $\mathcal{P}_s(\ell_1)$ coincides with $\delta_{(x,\lambda)}$ for some $\lambda \in \mathbb{C}$ and x satisfying (3.15).*

Actually, thanks to [1, Theorem 1.3] such sequence x is unique up to permutation.

Theorem 3.51. *There is no continuous character of the form $\delta_{(v,\lambda)}$ in the space $\mathcal{M}_{bs}(\ell_1)$, where*

$$v = \left\{c_1, \frac{c_2}{2}, \dots, \frac{c_n}{n}, \dots\right\},$$

and $|c_k| = 1$ for each k.

Proof. Assume otherwise, i.e., $\delta_{(v,\lambda)}$ is the restriction of some $\varphi \in \mathcal{M}_{bs}(\ell_1)$. Then by (3.13),

$$(\varphi \Diamond \varphi)(F_k) = \varphi(F_k)^2 = \left(\sum_{n=1}^{\infty} v_n^k\right)^2 = \left(\sum_{n=1}^{\infty} v_n^k\right)\left(\sum_{m=1}^{\infty} v_m^k\right) = \sum_{n,m=1}^{\infty} (v_n v_m)^k.$$

Therefore the sequence $(v_n v_m)_{n,m} = v \diamond v := s$, is, up to permutation, the one appearing in Proposition 3.50, so it must satisfy condition (3.15), that is, the sequence of the inverses has finite upper density.

Denote by $d(m)$ the number of divisors of a positive integer m. Then in the sequence $|s|$ of absolute values each number with absolute value $1/m$ can be found $d(m)$ times. So $|s|$ can be rearranged, if necessary, in the form

$$\left(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \dots, \underbrace{\frac{1}{m}, \dots \frac{1}{m}}_{d(m)}, \dots\right).$$

In particular, the index of the last entry of the element with absolute value $\frac{1}{m}$ is $\sum_{n=1}^{m} d(n)$. Hence for the sequence of the inverses and their counting number $\mathbf{n}(m)$, we have $\mathbf{n}(m) = \sum_{n=1}^{m} d(n)$. From Number Theory [2, Theorem 3.3] it is known that

$$\sum_{n=1}^{m} d(n) = m \ln m + 2(\gamma - 1)m + O(\sqrt{m}),$$

where γ is the Euler constant. So we are led to a contradiction because

$$\limsup_{m\to\infty} \frac{\mathbf{n}(m)}{m} \geq \limsup_{m\to\infty} \frac{m \ln m}{m} = \limsup_{m\to\infty} \ln m = \infty.$$

$\square$

Corollary 3.52. *There is a function of exponential type $g(t)$ for which there is no character $\varphi \in \mathcal{M}_{bs}(\ell_1)$ such that $\mathcal{G}(\varphi)(t) = g(t)$.*

Proof. It is enough to take a function of exponential (finite) type whose zeros are the elements of the sequence

$$\left\{\frac{1}{v_n}\right\} = \{-1, 2, \ldots (-1)^n n, \ldots\}.$$

Such is, for example, the function

$$g(t) = \prod_1^\infty \left(1 + (-1)^n \frac{t}{n}\right) \exp\left((-1)^n \frac{t}{n}\right).$$

□

Every $\varphi \in \mathcal{M}_{bs}(\ell_1)$ is determined by the sequence $(\varphi(F_m))$, that verifies the inequality $\limsup_n |\varphi(F_m)|^{1/m} \leq R(\varphi)$ because $\|F_m\| \leq 1$. As a byproduct of Theorem 3.51, we notice that the condition $\limsup_m |a_m|^{1/m} < +\infty$, does not guarantee that there is $\varphi \in \mathcal{M}_{bs}(\ell_1)$ such that $\varphi(F_m) = a_m$: Indeed, let $a_m = \sum_n \frac{1}{n^m}$ for $m > 1$ and arbitrary a_1. Then the sequence (a_m) is bounded, so $\limsup_m |a_m|^{1/m} \leq 1$, and if there existed $\varphi \in \mathcal{M}_{bs}(\ell_1)$ such that $\varphi(F_m) = a_m$, it would mean that for the sequence $x := (\frac{1}{n})$, $\varphi(F_m) = \sum_n \frac{1}{n^m}$, so $\delta_{(x,a_1)} = \varphi|_{P_s(\ell_1)}$.

Question 3.53. Can each element of $\mathcal{M}_{bs}(\ell_1)$ be represented as an entire function of exponential type with zeros $\{a_n\}_{n=1}^\infty$ such that either $\{a_n\} = \emptyset$ or $\sum_{n=1}^\infty \frac{1}{|a_n|} < \infty$?

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Чернега I.B. Симетричні поліноми і голоморфні функції на нескінченновимірних просторах. *Журнал Прикарпатського університету імені Василя Стефаника,* **2** (4) (2015), 23-49.

Стаття містить огляд основних результатів про спектри алгебр симетричних голоморфних функцій і алгебр симетричних аналітичних функцій обмеженого типу на банахових просторах.

Ключові слова: поліноми і аналітичні функції на банахових просторах, симетричні поліноми, спектри алгебр.

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CONVERGENCE INVESTIGATION OF ITERATIVE AGGREGATION METHODS FOR LINEAR EQUATIONS IN A BANACH SPACE

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Abstract. The sufficient conditions of convergence for a class of multi-parameter iterative aggregation methods are established. These conditions do not contain the requirements of positivity for the operators and aggregating functionals. Moreover, it is not necessary that the corresponding linear continuous operators are compressing.

Keywords: decomposition, parallelization computation methods, iterative aggregation.

1. INTRODUCTION

Problems of the operator equations decomposition are still actual. It is caused by the necessity of construction parallelization computation methods. Multi-parameter iterative aggregation is an effective method for decomposition of the high dimension problems (see [1]).

Let E be a Banach space and $A : E \longmapsto E$ be a linear continuous operator. Consider the equation

$$x = Ax + b, \qquad b \in E. \tag{1.1}$$

For such equations often it is assumed that: 1) the normal cone $K \subset E$ of positive elements is given; 2) semiordering in E is introduced by such elements; 3) compression operator A and element b are positive (see, for example, [2]–[4]). These and other requirements are caused by the specificity of the corresponding problems (see, for example, [5]–[10]). More detailed results for one-parametric method are given in [2, p. 155–158] and can be described by the formula

$$x^{(n+1)} = \frac{(\varphi, b)}{(\varphi, x^{(n)} - Ax^{(n)})} Ax^{(n)} + b \qquad (n = 0, 1, \ldots). \tag{1.2}$$

Here (φ, x) denotes the value of a linear functional $\varphi \in K^*$ on the elements $x \in E$, where K^* is a cone of positive elements in a dual Banach space E^*. The algorithm (1.2) is investigated in [2, p. 155–158] with the following assumptions: (i) A is a focusing operator [2, p. 77]; (ii) spectral radius $\rho(A)$ of the operator A is less than one; (iii) the functional φ is admissible. A functional φ is called an admissible if there exists a functional $g \in K^*$ such that $\varphi = A^* g$ and $(g, x) > (\varphi, x)$ for $x \in K$, $x \neq \Theta$, where A^* is conjugate to A operator and Θ is zero element in E.

In particular, if (1.1) is a system of linear algebraic equations with a matrix $A = \{a_{ij}\}$, then the focusing condition is valid when all a_{ij} are strictly positive numbers. For the linear integral operator of the following form

$$Ax = \int_a^b G(t,s)x(s)ds$$

the focusing condition is valid if the continuous function $G(t,s)$ satisfies the condition $G(t,s) \geqslant \varepsilon > 0$ for $t, s \in [a, b]$.

In [2, p. 158] it is noted that the theory of methods for iterative aggregation is not well developed and the conditions of their convergence are unknown. In particular, as it is indicated by numerous examples (see [2, p. 158]), one parametric method (1.2) often converges when the above conditions are not fulfilled.

In this work we investigate the multi-parameter algorithms of iterative aggregation using the methodology described in [11]–[15]. The established sufficient conditions of convergence do not contain the requirement of type $\rho(A) < 1$ for a spectral radius $\rho(A)$ of an operator A and condition of signs constancy for the operator A and of the aggregating functionals.

2. Construction of the Aggregative-Iterative Algorithm

We consider the equation (1.1) in a Banach space E. We do not need semiordering in E. Let the equation (1.1) is presented in the form

$$x = \sum_{j=1}^{N} A_j x + A_0 x + b, \tag{2.1}$$

where $A_0 : E \to E$, $A_j : E \to E$ $(j = 1, \dots, N)$, $b \in E$. Set the linear continuous functionals $\varphi^{(i)}$ $(i = 0, 1, \dots, N)$. Let us join to the equation (2.1) the auxiliary system of equations

$$y_i = \sum_{j=1}^{N} \lambda_{ij} y_j + (\varphi^{(i)}, B_i x) - (\varphi^{(i)}, b) \qquad (i = 0, 1, \dots, N). \tag{2.2}$$

Our basic assumptions are the following.

A) The equalities

$$(\varphi^{(i)}, A_j x) = \lambda_{ij} (\varphi^{(j)}, x) \qquad (j = 1, \dots, N;\ i = 0, 1, \dots, N) \tag{2.3}$$

hold.

B) There exists the operators $B_i : E \to E$ $(i = 0, 1, \dots, N)$ such that

$$(\varphi^{(i)}, A_0 x + B_i x) = \lambda_{i0} (\varphi^{(0)}, x) \qquad (i = 0, 1, \dots, N). \tag{2.4}$$

Let us construct the iterative process by the formulas

$$x^{(n+1)} = \sum_{j=1}^{N} A_j x^{(n)} + A_0 x^{(n)} + \sum_{j=1}^{N} a_j^{(n)} (y_j^{(n)} - y_j^{(n+1)}) + a_0^{(n)} (y_0^{(n)} - y_0^{(n+1)}) + b, \tag{2.5}$$

$$\begin{aligned} y_i^{(n+1)} &= \sum_{j=1}^{N} \lambda_{ij} y_j^{(n+1)} + \lambda_{i0} y_0^{(n+1)} + (\varphi^{(i)}, B_i x^{(n)}) + \sum_{j=1}^{N} \alpha_{ij}^{(n)} (y_j^{(n)} - y_j^{(n+1)}) \\ &+ \alpha_{i0}^{(n)} (y_0^{(n)} - y_0^{(n+1)}) - (\varphi^{(i)}, b) \qquad (i = 0, 1, \ldots, N), \end{aligned} \tag{2.6}$$

where the elements $a_j^{(n)} = a_j(x^{(n)}) \in E$ and real numbers $\alpha_{ij}^{(n)} = \alpha_{ij}(x^{(n)})$ satisfy the condition:

$$(\varphi^{(i)}, a_j(x)) + \lambda_{ij}(x) = \lambda_{ij} \qquad (x \in E,\ i, j = 0, 1, \ldots, N) \tag{2.7}$$

where λ_{ij} are real numbers and $a_j(x)$, $\alpha_{ij}(x)$ are continuous functions for $x \in E$.

3. Main Lemma

Let E' be an Euclid space of dimension $N + 1$. Consider the set of elements $x \in E$ and vectors $y = \{y_0, y_1, \ldots, y_N\}^T \in E'$, such that the equalities

$$(\varphi^{(i)}, x) + y_i = 0 \qquad (i = 0, 1, \ldots, N) \tag{3.1}$$

hold. Denote this set by ε_0. It is clear that ε_0 is a subspace of the space $\widetilde{E} = E \times E'$ equipped by the norm

$$\|(x, y)\| = \sqrt{\|x\|^2 + |y|^2},$$

where $\|x\|$ is the norm of an element $x \in E$ and $|y|$ is the Euclidean norm of a vector $y \in E'$.

Lemma 3.1. *Let the conditions A) and B) be satisfied. Let the matrix $I - \Lambda$ be nondegenerate, where I is the unit matrix in E' and*

$$\Lambda = \{\lambda_{ij}\} \qquad (i, j = 0, 1, \ldots, N). \tag{3.2}$$

Then solution $\{x^*, y^*\} \in \widetilde{E}$ of the system (2.1), (2.2) *belongs to ε_0, i.e. $\{x^*, y^*\} \in \varepsilon_0$.*

Proof. From the formulas (2.1)–(2.4) for $x = x^*$, $y_i = y_i^*$ we have

$$\begin{aligned} (\varphi^{(i)}, x^*) + y_i^* &= \sum_{j=1}^{N} (\varphi^{(i)}, A_j x^*) + (\varphi^{(i)}, A_0 x^*) + (\varphi^{(i)}, b) \\ &+ \sum_{j=1}^{N} \lambda_{ij} y_j^* + (\varphi^{(i)}, B_i x^*) - (\varphi^{(i)}, b) \\ &= \sum_{j=1}^{N} \lambda_{ij} \left[(\varphi^{(j)}, x^*) + y_j^* \right] + \left[(\varphi^{(i)}, A_0 x^* + B_i x^*) + \lambda_{i0} y_0^* \right] \\ &= \sum_{j=1}^{N} \lambda_{ij} \left[(\varphi^{(j)}, x^*) + y_j^* \right] + \lambda_{i0} \left[(\varphi^{(0)}, x^*) + y_0^* \right] \qquad (i = 0, 1, \ldots, N). \end{aligned}$$

Note, that the matrix $I - \Lambda$ is nondegenerate, so the lemma is proved.

Lemma 3.2. *Let the conditions A), B) and* (2.7) *be satisfied. If the matrix* $I - \Lambda$ *is nondegenerate and* $\{x^{(0)}, y^{(0)}\} \in \varepsilon_0$, *then* $\{x^{(n)}, y^{(n)}\} \in \varepsilon_0$ *for* $n = 0, 1, \ldots$

Proof. From the equalities (2.5)–(2.7) we have

$$\begin{aligned}
(\varphi^{(i)}, x^{(n+1)}) + y_i^{(n+1)} =& \sum_{j=1}^{N} (\varphi^{(i)}, A_j x^{(n)}) + (\varphi^{(i)}, A_0 x^{(n)}) + \sum_{j=1}^{N} (\varphi^{(i)}, a_j^{(n)})(y_j^{(n)} - y_j^{(n+1)}) \\
&+ (\varphi^{(i)}, a_0^{(n)})(y_0^{(n)} - y_0^{(n+1)}) + (\varphi^{(i)}, b) + \sum_{j=1}^{N} \lambda_{ij} y_j^{(n+1)} + \lambda_{i0} y_0^{(n+1)} + (\varphi^{(1)}, B_0 x^{(n)}) \\
&+ \sum_{j=1}^{N} \alpha_{ij}^{(n)} (y_j^{(n)} - y_j^{(n+1)}) + \alpha_{i0}^{(n)} (y_0^{(n)} - y_j^{(n+1)}) - (\varphi^{(i)}, b) = \sum_{j=1}^{N} \lambda_{ij} (\varphi^{(j)}, x^{(n)}) \\
&+ (\varphi^{(i)}, A_0 x^{(n)} + B_i x^{(n)}) + \left[(\varphi^{(i)}, a_0^{(n)}) + \alpha_{i0}^{(n)} \right] y_0^{(n)} + \sum_{j=1}^{N} \left[(\varphi^{(i)}, a_j^{(n)}) + \alpha_{ij}^{(n)} \right] y_j^{(n)} \\
&+ \left[\lambda_{i0} - (\varphi^{(i)}, a_0^{(n)}) + \alpha_{i0}^{(n)} \right] y_0^{(n+1)} + \left[\lambda_{ij} - (\varphi^{(i)}, a_j^{(n)}) - \alpha_{ij}^{(n)} \right] y_j^{(n+1)} \\
=& \sum_{j=1}^{N} \lambda_{ij} \left[(\varphi^{(j)}, x^{(n)}) + y_j^{(n)} \right] + \lambda_{i0} \left[(\varphi^{(0)}, x^{(n)}) + y_0^{(n)} \right] \qquad (i = 0, 1, \ldots, N).
\end{aligned} \tag{3.3}$$

Since $\{x^{(0)}, y^{(0)}\} \in \varepsilon_0$, equalities (3.3) are the reason for using of the induction principle. The proof is complete.

From these two lemmas we obtain the following assertion.

Lemma 3.3. *Let the conditions A), B) and* (2.7) *be satisfied. If there exists the matrix* $(I - \Lambda)^{-1}$, $\{x^{(0)}, y^{(0)}\} \in \varepsilon_0$, *and* $\{x^*, y^*\}$ *is the solution of the system* (2.1), (2.2) *in* $\widetilde{E}$, *then*

$$(\varphi^{(i)}, x^{(n)} - x^*) + y_i^{(n)} - y_i^* = 0 \qquad (i = 0, 1, \ldots, N;\ n = 0, 1, \ldots). \tag{3.4}$$

Proof. It is enough to note that (3.4) is a consequence of the equalities (3.1) for $\{x^{(0)}, y^{(0)}\}$ and $\{x^*, y^*\}$.

4. Sufficient Conditions for the Convergence of the Algorithm (2.5), (2.6)

Denote $a^{(n)} = \{a_0^{(n)}, a_1^{(n)}, \ldots, a_N^{(n)}\}$, $[\varphi, b] = \{(\varphi^{(0)}, b), (\varphi^{(1)}, b), \ldots, (\varphi^{(N)}, b)\}^T$, $[\varphi, Bx] = \{(\varphi^{(0)}, B_0 x), (\varphi^{(1)}, B_1 x), \ldots, (\varphi^{(N)}, B_N x)\}^T$. Let us rewrite the formulas (2.2), (2.5), (2.6) in the form

$$y = \Lambda y + [\varphi, Bx] - [\varphi, b], \tag{4.1}$$

$$x^{(n+1)} = A x^{(n)} + a^{(n)} (y^{(n)} - y^{(n+1)}) + b, \tag{4.2}$$

$$y^{(n+1)} = \Lambda y^{(n+1)} + [\varphi, B x^{(n)}] + \alpha^{(n)} (y^{(n)} - y^{(n+1)}) - [\varphi, b], \tag{4.3}$$

respectively, where the matrix Λ is defined by (3.2).

From the formulas (1.1), (4.1) and (4.2), (4.3) we obtain

$$x^{(n+1)} - x^* = A(x^{(n)} - x^*) + a^{(n)} (y^{(n)} - y^*) - a^{(n)} (y^{(n+1)} - y^*), \tag{4.4}$$

$$y^{(n+1)} - y^* = (\Lambda - \alpha^{(n)})(y^{(n+1)} - y^*) + \alpha^{(n)}(y^{(n)} - y^*) + [\varphi, B(x^{(n)} - x^*)]. \tag{4.5}$$

Hence

$$y^{(n+1)} - y^* = (I - \Lambda + \alpha^{(n)})^{-1}\alpha^{(n)}(y^{(n)} - y^*) + (I - \Lambda + \alpha^{(n)})^{-1}[\varphi, B(x^{(n)} - x^*)]. \tag{4.6}$$

This equality together with (4.4) imply the equality

$$\begin{aligned} x^{(n+1)} - x^* =& A(x^{(n)} - x^*) + a^{(n)}(y^{(n)} - y^*) - a^{(n)}(I - \Lambda + \alpha^{(n)})^{-1}\alpha^{(n)}(y^{(n)} - y^*) - \\ & - a^{(n)}(I - \Lambda + \alpha^{(n)})^{-1}[\varphi, B(x^{(n)} - x^*)]. \end{aligned} \tag{4.7}$$

Therefore, using (4.7) and (3.4), we obtain

$$x^{(n+1)} - x^* = A(x^{(n)} - x^*) - a^{(n)}(I - \Lambda + \alpha^{(n)})^{-1}((I - \Lambda)[\varphi, x^{(n)} - x^*] + [\varphi, B(x^{(n)} - x^*)]). \tag{4.8}$$

From (4.6), (4.8) it follows the next assertion.

Theorem 4.1. *Let the conditions of Lemma 3.3 be satisfied. Let the operator, generated by the right part of the equalities* (4.6), (4.7) *with respect to the pair* $\{x - x^*, y - y^*\}$ *with* $(x, y) \in \varepsilon_0$*, be compression. This means that the operator* $H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$ *is compression with respect to the pair* $\{w, z\}$*, where* $w \in E$*,* $z \in E'$*,* $\{w, z\} \in \varepsilon_0$*,*

$$h_{11}w = Aw + a(x)(I - \Lambda + \alpha(x))^{-1}[\varphi, Bw],$$

$$h_{12}z = a(x)(I - \Lambda + \alpha(x))^{-1}(I - \Lambda)z,$$

$$h_{21}w = (I - \Lambda + \alpha(x))^{-1}[\varphi, Bw],$$

$$h_{22}z = (I - \Lambda + \alpha(x))^{-1}\alpha(x)z.$$

Then a sequence $\{x^{(n)}\}$*, obtained by the algorithm* (4.2), (4.3) *converges to the solution* $x^* \in E$ *of the equation* (1.1) *not slower than a geometric progression with common ratio* $q < 1$*, where* q *is a norm of the operator* H *in the space* $\widetilde{E}$*.*

From the Theorem 4.1 we can get next proposition.

Theorem 4.2. *Let the conditions of Lemma 3.3 be satisfied. Define the operator* H_0 *by the formula*

$$H_0 w = Aw - a(x)(I - \Lambda + \alpha(x))^{-1}((I - \Lambda)[\varphi, w] + [\varphi, Bw]). \tag{4.9}$$

If for $(x, y) \in \varepsilon_0$ *the operator* H_0 *is compression with a compression constant* $q_0 < 1$*, then a sequence* $\{x^{(n)}\}$*, obtained by* (4.2), (4.3)*, converges to the solution* x^* *of the equation* (1.1) *not slower than a geometric progression with common ratio* q_0*.*

Proof. Rewrite the equalities (3.4) in the form

$$[\varphi, x^{(n)} - x^*] + (y^{(n)} - y^*)^T = \Theta, \tag{4.10}$$

where Θ is zero column vector. From (4.9), (4.10) we obtain that the theorem is proved.

5. Multi-Parameter Iterative Aggregation

Define elements $a_j(x)$ by the formula

$$a_j(x) = \frac{A_j x}{(\varphi^{(j)}, x)} \qquad (j = 0, 1, \ldots, N,\ x \in E). \tag{5.1}$$

In this case the algorithm (2.5), (2.6) can be defined by the interpolation formula

$$x^{(n+1)} = \sum_{j=1}^{N} \frac{(\varphi^{(j)}, x^{(n+1)})}{(\varphi^{(j)}, x^{(n)})} A_j x^{(n)} + b + \frac{(\varphi^{(0)}, x^{(n+1)})}{(\varphi^{(0)}, x^{(n)})}. \tag{5.2}$$

This algorithm is an analogue of the method (19.12), (19.13) from [2, p.156]. From the nondegeneracy of the matrices $I - \Lambda$, $I - \Lambda + \alpha(x)$ for $\{x, y\} \in \varepsilon_0$ ($x \in E$, $y \in E'$) it follows that we can choose the aggregation functionals $\varphi^{(i)}$, matrices $\Lambda = \{\lambda_{ij}\}$ and $\alpha(x) = \{\alpha_{ij}(x)\}$, which are used in described above methodology.

If $\alpha(x)$ is a zero matrix, then the algorithm (2.5), (2.6) does not converted to one of the projection-iterative methods, that are investigated in [16, 17].

It is also possible to construct other multi-parameter algorithms of iterative aggregation. For example,

$$x^{(n+1)} = A_0 x^{(n)} + \sum_{j=1}^{N} \frac{(\varphi^{(j)}, x^{(n+1)})}{(\varphi^{(j)}, x^{(n)})} A_j x^{(n)} + b. \tag{5.3}$$

Let us consider the case, when we use the formulas

$$x^{(n+1)} = \sum_{j=1}^{N} A_j x^{(n)} + \sum_{j=1}^{N} a_j^{(n)} (y_j^{(n)} - y_j^{(n+1)}) + A_0 x^{(n)} + b, \tag{5.4}$$

$$y^{(n+1)} = \Lambda y^{(n+1)} + [\varphi, B x^{(n)}] + \alpha^{(n)} (y^{(n)} - y^{(n+1)}) - [\varphi, A_0 x^{(n)}] - [\varphi, b] \tag{5.5}$$

instead of the formulas (2.5), (2.6) respectively.

Everywhere in the formulas (5.3)–(5.5) all indices i, j take values from 1 to N, i.e. $i \neq 0$ and $j \neq 0$.

For the algorithm (5.4), (5.5) we remain the structure of the matrix H and of the set ε_0. In this case we have

$$h_{11} w = A w + a(x)(I - \Lambda + \alpha(x))^{-1} [\varphi, (B - A_0) w],$$

$$h_{12} z = a(x)(I - \Lambda + \alpha(x))^{-1} (I - \Lambda) z,$$

$$h_{21} w = (I - \Lambda + \alpha(x))^{-1} [\varphi, (B - A_0) w],$$

$$h_{22} z = (I - \Lambda + \alpha(x))^{-1} \alpha(x) z.$$

In these circumstances, the Theorem 4.2 is still valid for the algorithm (5.4), (5.5).

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Копач М.І., Обшта А.Ф., Шувар Б.А. Дослідження збіжності методів ітеративного агрегування для лінійних рівнянь в банаховому просторі. *Журнал Прикарпатського університету імені Василя Стефаника,* **2** (4) (2015), 50-57.

У роботі встановленні достатні умови збіжності одного класу багатопараметричних агрегаційно-ітеративних методів. Отримані результати не містять вимог про додатність операторів і агрегуючих функціоналів, а також не потребують, щоб відповідні лінійні оператори були стискуючими.

Ключові слова: декомпозиція, розпаралелення обчислень, ітеративне агрегування.

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ON GENERALIZATIONS OF THE HILBERT NULLSTELLENSATZ FOR INFINITY DIMENSIONS (A SURVEY)

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Abstract. The paper contains a proof of Hilbert Nullstellensatz for the polynomials on infinite-dimensional complex spaces and for a symmetric and a block-symmetric polynomials.

Keywords: polynomials, symmetric polynomials, block-symmetric polynomials, algebra of polynomials, Hilbert Nullstellensatz, algebraic basis.

1. INTRODUCTION

The Hilbert Nullstellensatz is a classical princip in Algebraic Geometry and actually its starting point. It provides a bijective correspondence between affine varieties, which are geometric objects and radical ideals in a polynomials ring, wich are algebraic objects. For the proof and applications of the Hilbert Nullstellensatz we refer the reader to [6].

The question whether a bounded polynomial functionalon a complex Banach space X is determined by its kernel the set of zeros under te assumption that all the factors of its decomposition into irreducible factors are simple was posed by Mazur and Orlich (see also Problem 27 in [10]). A positive answer to this question it follows from Theorem 2 of the present paper. Moreover, this result remains valid even when the ring of bounded polynomial functionals is replaced by any ring of polynomials for which there exists a decomposition into ireducible factors satisfying the following condition along with each polynomial $P(x)$ that it contains the ring also contains the polynomial $P_{\lambda;x_0}(x) = P(x_0 + \lambda x)$, where $x \in X$ and $\lambda \in \mathbb{C}$.

Let X and Y be vector spaces over the field $\mathbb{C}$ of complex numbers. A mapping $\overline{P}_k(x_1, \ldots, x_k)$ from the Cartesian product X^k into Y is $k - linear$ if it is linear in each component. The restriction P_k of the k-linear operator $\overline{P}_k$ to the diagonal $\triangle = \{(x_1, \ldots, x_k) \in X^k : x_1 = \ldots = x_k\}$, which can be naturally identified with X, is a homogeneous polynomial of degree k (briefly, a k-monomial). A finite sum of k-monomials, $0 \leq k \leq n$, $P(x) = P_0(x) + P_1(x) + \ldots + P_n(x)$, $P_n \neq 0$ is a polynomial of degree n. For general properties of polynomials on abstract linear spaces we refer the reader to [4].

This paper is devoted to generalizations of the Hilbert Nullstellensatz of infinite dimensional spaces. In Section 2 we consider the case of abstract infinite dimension complex linear spaces. Section 3 is devoted to continuous polynomials on complex Banach spaces. In Section 4 we examin symmetric polynomials on ℓ_p and Section 5 contains some new results about Nullstellensatz for block-symmetric polynomials.

2. The Nullstellensatz on Infinite-Dimensional Complex Spaces

All results of this section are proved in [15].

Let us denote by X a complex vector space, by $\mathcal{P}(X)$ the algebra of all complex-valued polynomials on X. Let $\mathcal{P}_0(X)$ be a subalgebra of $\mathcal{P}(X)$ satisfying the following conditions:

(1) If $P(x) \in \mathcal{P}_0(X)$, then $P_{x_0;\lambda}(x) = P(\lambda x + x_0) \in \mathcal{P}_0(X)$ for any $x_0 \in X$ and $\lambda \in \mathbb{C}$.
(2) If $P \in \mathcal{P}_0(X)$, $P = P_1 P_2$; $P_1 \neq 0, P_2 \neq 0$, then $P_1 \in \mathcal{P}_0(X)$ and $P_2 \in \mathcal{P}_0(X)$.

That is, the algebra $\mathcal{P}_0(X)$ is factorial and closed under translation. We shall agree to call such algebras of polynomials *FT-algebra*.

It is obvious that $\mathcal{P}(X)$ is an FT-algebra. A typical example of an FT-algebra is algebra of bounded polynomials (on bounded subset) on a locally convex space X. We shall denote this algebra by $\mathcal{P}_b(X)$. Anothe example of an FT-algebra is provided by the polynomials formed by finite sums of finite products of continuous linear functionals on X (polynomials of finite type). If Y is subspace of X, we take $\mathcal{P}_0(Y)$ to mean the restrictions of the polynomials of $\mathcal{P}_0(X)$ to Y. It easy to see that $\mathcal{P}_b(Y)$ coincides with the algebra of bounded polynomials on Y.

Let $P_\gamma(x) \in \mathcal{P}_0(X)$ be a family of polynomials, where γ belongs to an index set Γ. We recall that an ideal (P_γ) in $\mathcal{P}_0(X)$ is a set $J = \left\{ P \in \mathcal{P}_0(X) : P = \sum\limits_{\gamma\in\Gamma} Q_\gamma(x) P_\gamma(x), Q_\gamma \in \mathcal{P}_0(X) \right\}$, where the sum $\sum\limits_{\gamma\in\Gamma} Q_\gamma(x)P_\gamma(x)$ contains only a finite number of terms that are not identifically zero. A linearly independent subset $\{P_{\gamma_\beta}\}$ of the set $\{P_\gamma\}$ such that $(P_\gamma) = (P_{\gamma_\beta})$ is a linear basis of the ideal J. For an ideal $J \in \mathcal{P}_0(X)$, $V(J)$ denotes the *zero* of J, that is, the common set of zeros of all polynomials in J. Let G be a subset of X. Then $I(G)$ denotes the *hull* of G, that is, a set of all polynomials in $\mathcal{P}_0(X)$ which vanish on G. The set $radJ$ is called the *radical* of J if $P^k \in J$ for some positive integer k implies $P \in radJ$. P is called a *radical* polynomial if it can be represented by a product of mutually different irreducible polynomials. In the case $(P) = rad(P)$.

It is easy to see that $I(G)$ is an ideal in $\mathcal{P}_0(X)$. The main problem that we shall solve consists of establising conditions under wich the equality

$$I(V(J)) = J$$

holds for the ideal $J \in \mathcal{P}_0(X)$ that is, an ideal in $\mathcal{P}_0(X)$ is uniquely determined by its set of zeros.

In the finite-dimensional case the answer to this question is provided by the Hilbert Nullstellensatz, which asserts that a necessary and sufficient condition for this to happen is that the ideal J be equal to its radical (which we shall define below). We remark that for the infinite-dimensional case this condition is not sufficient. (A counterexample will be given).

Lemma 2.1. *Let $P_1, \ldots, P_n$ be polynomials on X and $\deg P_1 \geq \deg P_2 \geq \ldots \geq \deg P_n > 0$. Then there exists an element $h \in X$ such that for any $x \in X$ the degree of the scalar-valued polynomial $P_1(x+th)$ in t is $\deg P_1$, and the polynomials $P_2, \ldots, P_n$ depend on h, that is, for each $P_i, i = 2, \ldots, n$, there exists x_i such that the scalar-valued polynomial $P_i(x_i + th)$ in t is of positive degree.*

Proof. For $n = 1$ the assertion of the lemma is obvious. Assume it is true for $n-1$. Let h_1 be the required element for $P_1, \ldots, P_{n-1}$. Assume that P_n is independent of h_1, that is, $P_n(x + th_1) = P_n(x)\ \forall x \in X$. Let h_2 be an elementof X such that P_n depends on h_2. We make the definition $h(\lambda) := h_1 + \lambda h_2, \lambda \in \mathbb{C}$. Consider the family of scalar-valued polynomials $P_1(x + th(\lambda))$ in t with parameters λ, x. For any x there is only a finite set of λ, at which the polynomial $P_1(x + th(\lambda))$ is of degree less than $\deg P_1$ in t.

Indeed, let $\deg P_1 = m$, and let $P_1 = \sum\limits_{i=0}^{m} f_i$ be an expansion in monomials. Then $P_1(x + th(\lambda))$ can be given in the following form:

$$\begin{aligned} P_1(x + th(\lambda)) &= \sum_{i=1}^{m} f_i(x + th(\lambda)) = \sum_{i=1}^{m} t^j \bar{f}_i(x, \ldots, x, \overbrace{h(\lambda), \ldots, h(\lambda)}^{j}) \\ &= t^m f_m(h(\lambda)) + \sum_{k<m} \sum_{j \leq k} t^j q_j(x + h(\lambda)), \end{aligned}$$

where $\bar{f}_i$ are $i-$ linear forms corresponding to the monomials f_i;

$$q_j = \sum_i \bar{f}_i(x, \ldots, x, \overbrace{h(\lambda), \ldots, h(\lambda)}^{j}).$$

If $\deg P_1(x + th(\lambda')) < m$ for some value λ' of the parameter λ, then $f_m(h(\lambda')) = f_m(h_1 + \lambda' h_2) = 0$. But, since $f_m(h_1 + \lambda h_2)$ is polynomial in the variable λ (for fixed h_1 and h_2), it can have only a finite number of zeros without being identically zero. Assume that $f_m(h_1 + \lambda h_2) \equiv 0$. Then this relation also holds for $\lambda = 0$. Hence $\deg P_1(x + th(0)) = \deg P_1(x + th_1) < m$, which contradicts the choice of h_1.

Similarly, for each $i = 2, \ldots, n-1$ there exists a finite set of values of the parameter λ at which the polynomials $P_i(x + th(\lambda))$ have smaller degree in t than $\deg P_i$, in particular, degree 0. Hence there exists a number $\lambda_0 \neq 0$ such that $\deg P_1(x + th(\lambda_0)) = m$ with respect to t, and the polynomials P_i depend on $h(\lambda_0)$ for $1 < i < n$. Moreover, P_n also depends on $h(\lambda_0)$, since $P_n(x + th(\lambda_0)) = P_n(x + t\lambda_0 h_2)$. Therefore, $h := h(\lambda_0)$ is the required element for n. The lemma is now proved. □

Theorem 2.2. *Let X be a complex vector space of arbitrary (possibly infinite) dimension, and let $P_1(x), \ldots, P_n(x) \in \mathcal{P}_0(X)$, where $\mathcal{P}_0(X)$ is an FT-algebra. Then there exists an element $h \in X$, a subspace Z complementary to $\mathbb{C}h$ in X, and polynomial functionals $G_1, \ldots, G_{n-1} \in \mathcal{P}_0(X)$ such that:*

(1) $g_k(z + th) = g_k(z)\ \forall z \in Z, t \in \mathbb{C}, k = 1, \ldots, n-1$.
(2) *All G_k belong to the ideal $(P_1, \ldots, P_n)$ in the algebra $\mathcal{P}_0(X)$.*
(3) *The set of zeros of the ideal $(g_1, \ldots, g_{n-1})$ in the algebra $\mathcal{P}_0(Z)$ is the projection of the zeros of the ideal $(P_1, \ldots, P_n)$ in $\mathcal{P}_0(X)$ onto the subspace Z along h.*
(4) *If $g_k \equiv 0, k = 1, \ldots, n-1$, then $P_1, \ldots, P_n$ have a common divisor.*

Proof. Let $\deg P_1 = \max_i \deg P_i$ and let $h \in X$ be an element such that the degree of the polynomial $P_1(x+th)$ in the variable $t \in \mathbb{C}$ equals $\deg P_1$ for all $x \in X$ and the polynomials $P_1, \ldots, P_n$ depend on h. Such an element exists in accordance with Lemma 2.1. Concider the polynomials $P_1, \ldots, P_n$ as elements of the algebra $(\mathcal{P}_0(Z))[t]$, where Z is a closed subspace complementary to $\mathbb{C}h$ in X. That is, the elements of the algebra $(\mathcal{P}_0(Z))[t]$ are polynomials of t with coefficients in the fieldof quotients of elements of $\mathcal{P}_0(Z)$. We shall denote them by $\tilde{P}_1(t), \ldots, \tilde{P}_n(t)$ respectively. We may assume that $\deg \tilde{P}_1(t) \geq \tilde{P}_2(t) \geq \ldots \geq \tilde{P}_n(t)$. Division with remainder holds in the algebra $(\mathcal{P}_0(Z))[t]$. Therefore for $\tilde{P}_1(t)$ and $\tilde{P}_2(t)$ there exist $P_2^1(t)$ and $P_2^1(t)$ in $(\mathcal{P}_0(Z))[t]$ such that

$$\tilde{P}_1 - Q_2^1 \tilde{P}_2 = P_2^1. \tag{2.1}$$

If $\deg P_2^1 \geq \deg \tilde{P}_3$, there exist Q_3^1 and P_3^1 in $(\mathcal{P}_0(Z))[t]$ such that

$$P_2^1 - Q_3^1 \tilde{P}_3 = P_3^1. \tag{2.2}$$

When $\deg P_2^1 < \deg \tilde{P}_3$, we set $Q_3^1 = 0, P_3^1 = P_2^1$. Continuing this process, we obtain the following relations:

$$\begin{aligned} &P_3^1 - Q_4^1 \tilde{P}_4 = P_4^1, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{2.3}$$

$$P_{n-1}^1 - Q_n^1 \tilde{P}_n = \tilde{P}_{n+1}, \tag{2.4}$$

where $\deg \tilde{P}_{n+1} < \deg \tilde{P}_n$. From relations (2.1)-(2.4), we have:

$$\tilde{P}_1 - \sum_{i=2}^{n} Q_i^1 \tilde{P}_i = \tilde{P}_{n+1}.$$

For the elements $\tilde{P}_2, \ldots, \tilde{P}_{n+1} \in (\mathcal{P}_0(Z))[t]$ we obtain similarly the relations

$$\tilde{P}_2 - \sum_{i=3}^{n+1} Q_i^2 \tilde{P}_i = \tilde{P}_{n+2},$$

$\deg \tilde{P}_{n+2} < \deg \tilde{P}_{n+1}$; for $\tilde{P}_3, \ldots, \tilde{P}_{n+2}$:

$$\tilde{P}_3 - \sum_{i=4}^{n+2} Q_i^3 \tilde{P}_i = \tilde{P}_{n+3},$$

$\deg \tilde{P}_{n+3} < \deg \tilde{P}_{n+2}$, and so on.

Since the sequence $\deg \tilde{P}_{n+1}, \deg \tilde{P}_{n+2}, \ldots$ is strictly decreasing, by continuing this process we obtain for a coefficient k_1 :

$$\tilde{P}_{k_1-1} - \sum_{i=k_1}^{n+k_1-2} Q_i^{k_1} \tilde{P}_i = \tilde{P}_{n+k_1-1}.$$

Moreover, $\deg \tilde{P}_{n+k_1-1} = 0$, that is, $\tilde{P}_{n+k_1-1} \in \mathcal{P}_0(Z)$. We introduce the notation $G_1 = \tilde{P}_{n+k_1-1}$. Consider the elements $\tilde{P}_{k_1}(t), \ldots, \tilde{P}_{n+k_1-2}(t) \in (\mathcal{P}_0(Z))[t]$. There are $n-1$ of them, all depending on t. Applying the preceding reasoning to them, we obtain for some $k_2 > k_1$:

$$\tilde{P}_{k_2-1} - \sum_{i=k_2}^{n+k_2-3} Q_i^{k_2} \tilde{P}_i = \tilde{P}_{n+k_2-2},$$

where $\tilde{P}_{n+k_2-2} \in \mathcal{P}_0(Z)$, $\deg \tilde{P}_{n+k_2-2} < \deg \tilde{P}_{n+k_2-3} < \dots$. We introduce the notation $G_2 = \tilde{P}_{n+k_2-2}$. Consider the polynomials $\tilde{P}_{k_2}(t), \dots, \tilde{P}_{n+k_2-3}(t) \in (\mathcal{P}_0(Z))[t]$. There are $n-2$ of them, all dependent on t, and the preceding reasoning is applicable to them.

Thus at step r we obtain, for some $k_r > k_{r-1>\dots>k_1}$:

$$\tilde{P}_{k_r-1} - \sum_{i=k_r}^{n+k_r-r-2} Q_i^{k_r} \tilde{P}_i = \tilde{P}_{n+k_r-r-1},$$

where $\tilde{P}_{n+k_r-r-1} \in \mathcal{P}_0(Z)$. We introduce the notation $G_r = \tilde{P}_{n+k_r-r-1}$. At step $r = n-1$ our algorithm coincides with the Euclidean algorithm for the polynomials $\tilde{P}_{k_{n-1}}(t), \tilde{P}_{k_{n-1}+1}(t)$. That is, for some $k_{n-1} > \dots > k_1$ we find:

$$\tilde{P}_{k_{n-1}} - Q_{k_{n-1}+1}^{k_{n-1}} \tilde{P}_{k_{n-1}+1} = \tilde{P}_{k_{n-1}+2},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\tilde{P}_{k_n-4} - Q_{k_n-3}^{k_n-4} \tilde{P}_{k_n-3} = \tilde{P}_{k_n-2}, \tag{2.5}$$

$$\tilde{P}_{k_n-3} - Q_{k_n-2}^{k_n-3} \tilde{P}_{k_n-2} = \tilde{P}_{k_n-1}, \tag{2.6}$$

where $\tilde{P}_{k_n-1} \in \mathcal{P}_0(Z)$. We introduce the notation $G_{n-1} = \tilde{P}_{k_n-1}$.

It is clear from the algorithm that all the polynomials $\tilde{P}_i \in (\mathcal{P}_0(Z))[t]$ belong to the ideal $(\tilde{P}_1, \tilde{P}_2, \dots, \tilde{P}_n)$ in the algebra $(\mathcal{P}_0(Z))[t]$. In particular, this is true also for $G_r = \tilde{P}_{n+k_r-1}$. That is, there exist polynomials V_i^k, $k = 1, \dots, n-1$, $i = 1, \dots, n$, in the algebra $(\mathcal{P}_0(Z))[t]$ such that

$$\sum_{i=1}^{n} \tilde{P}_i V_i^k = G_k$$

for $k = 1, \dots, n-1$. Multiplying each of these equalities by the common denominator $a_k \in \mathcal{P}_0(Z)$ of the coefficients of the terms of degree t in $\mathcal{P}_0(Z)$ and passing to the algebra $\mathcal{P}_0(X)$, we find that there exist polynomials $v_i^k \in \mathcal{P}_0(X)$, such that

$$\sum_{i=1}^{n} P_i v_i^k = g_k, \tag{2.7}$$

where $g_k = G_k a_k$.

Thus we have found a sequence of polynomials $g_1, \dots, g_{n-1}$, that actually belong to $\mathcal{P}_0(Z)$, more precisely: $g_k(z+th) = g_k(z) \forall z \in Z$. In addition, all g_k belong to the ideal $(P_1, \dots, P_n)$. Let $z_0 \in Z$ be a common zero of the polynomials g_k. Then $z_0 + th$ is a common zero of g_k, $k = 1, \dots, n-1$, for any $t \in \mathbb{C}$. We multiply Eq. (2.6) by the common denominator $b_1 \in \mathcal{P}_0(Z)$ of the coefficients of the powers of t and pass to the algebra $\mathcal{P}_0(Z)$. Then,

$$P_{k_n-3} - q_{k_n-2}^{k_n-3} P_{k_n-2} = g_{n-1},$$

where $P_i = \tilde{P}_i b_1$, $q_i = Q_i b_1$. Therefore $P_{k_n-3}(z_0+th)$ is divisible by $P_{k_n-2}(z_0+th)$ (since $g_{n-1}(z_o+th) = 0$). Let us multiply Eq. (2.5) by b_2, the common denominator of the powers of t in (2.5), and substitute the value of P_{k_n-3} in place of P_{k_n-3} itself:

$$P_{k_n-4} - q_{k_n-3}^{k_n-4}(g_{n-1} + q_{k_n-2}^{k_n-3} P_{k_n-2}) - P_{k_n-2} = 0.$$

Taking account of the relation $g_{n-1}(z_0+th) = 0$, we find that $P_{k_n-4}(z_0+th)$ is divisible by $P_{k_n-2}(z_0+th)$. Working from bottom to top, we find that the polynomials $b(z_0+th)P_1(z_0+th), \dots, b(z_0+th)P_n(z_0+th)$ are divisible by $P_{k_n-2}(z_0+th)$, where b is polynomial in $\mathcal{P}_0(Z)$.

Assume that $P_{k_n-2}(z_0+th) \equiv const$ (with respect t). This means that the degree of the polynomial $P_{k_n-2}(z_0+th)$ is less than the degree of the polynomial $\tilde{P}_{k_n-2}(t) \in (\mathcal{P}_0(Z))[t]$, since by construction $\deg \tilde{P}_{k_n-2} > 0$. Then we also have $\deg P_{k_n-3}(z_0+th) < \deg \tilde{P}_{k_n-3}(t)$. Working from the bottom upward, we find that $\deg P_1(z_0+th)$, as a polynomial in t, is less than $\deg \tilde{P}_1 = \deg P_1$. But the equality $\deg \tilde{P}_1 = \deg P_1$ (which holds by the choice of h) means that the monomial of highest degree in t in the polynomial $P_1(z_0+th)$ is independent of $z \in Z$, so that this is impossible. Hence $P_{k_n-2}(z_0+th) \neq const$, and therefore, first of all, the fact that $b(z_0+th)P_i(z_0+th)$ is divisible by $P_{k_n-2}(z_0+th)$ for $1 \leq i \leq n$ implies that $P_i(z_0+th)$ is divisible by $P_{k_n-2}(z_0+th)$, $1 \leq i \leq n$, since b is independent of h and $P_{k_n-2}(z_0+th)$ depends on h; second there exists $t_0 \in \mathbb{C}$ such that $P_{k_n-2}(z_0+th) = 0$. Thus $x_0 = z_0 + t_0 h$ is a common zero of the polynomials $P_1, \ldots, P_n$.

As a result we have the following: if z_0 is a zero of the ideal $(g_1, \ldots, g_{n-1})$, then for some t_0 we find that $x_0 = z_0 + t_0$ is a zero of the ideal $(P_1, \ldots, P_n)$. It follows from Eqs. (2.7) that the converse is also true: every zero of the ideal $(P_1, \ldots, P_n)$ is a zero of the ideal $(g_1, \ldots, g_{n-1})$, and hence its projection of the zeros of the ideal $(P_1, \ldots, P_n)$ on the subspace Z along h. In the case when $g_k \equiv 0$ for all k we find that all P_i, $i = 1, \ldots, n$, are divisible by $P_{k_n-2}(z+th)$ for every $z \in Z$, that is, $(P_1(x), \ldots, P_n(x))$ have the common divisor $P_{k_n-2}(x)$. The theorem is now proved. □

Remark 1. In the case $\dim X = 1$ the proposed algorithm becomes the general Euclidean algorithm for finding a common divisor for n polynomials in one variable

Corollary 2.3. *Let $J = (P_1, \ldots, P_n)$ be an ideal of polynomials in $\mathcal{P}_0(X)$ and* $\dim X \geq n$. *Then there exist elements $h_1, \ldots, h_m \in X$, a subspace $W \subset X$ of codimension $m \leq n-1$, and a polynomial $f \in \mathcal{P}_0(X)$ such that:*

(1) *$f \in J$.*

(2) *f is independent of $h_1, \ldots, h_m$, that is, for any $w \in W$ $f(w + t_1 h_1 + \ldots + t_m h_m) = f(w)$, where $t_1, \ldots, t_n$ are arbitrary elements of $\mathbb{C}$.*

(3) *The kernel of f is the projection of the set $V(J)$ on W along the subspace $H_m = lin(h_1, \ldots, h_m)$.*

Proof. We apply Theorem 2.2 to the ideal $J = (P_1, \ldots, P_n)$. Let $g_1, \ldots, g_{n-1}$ be polynomials, h an element of X, Z is the subspace of X whose existence is guaranteed by the theorem. We revise the notation for $g_i^1 := g_i$, $i = 1, \ldots, n-1$, $h_1 := h$, $Z_1 = Z$. Applying Theorem 2.2 to the polynomials $g_1^1, \ldots, g_{n-1}^1$, we obtain polynomials $g_1^2, \ldots, g_{n-2}^2$, element $h_2 \in X$, and a subspace $Z_2 \subset X$. Here h_2 can be chosen from the subspace Z_1 and $Z_2 \subset Z_1$. Applying Theorem 2.2 several times at step $m \leq n-1$, we obtain a polynomial $g_1^m =: f \in \mathcal{P}_0(X)$ such that $f \in J$. Indeed

$$J = (P_1, \ldots, P_n) \supset (g_1^1, \ldots, g_{n-1}^1) \supset \ldots \supset (g_1^m) = (f), \quad f \in (f). \tag{2.8}$$

Let $w_0 \in \ker f$. Then by Theorem 2.2 we have $w_0 + t_m^0 \in V((g_1^{m-1}, g_2^{m-1}))$ for some t_m^0. Then $w_0 + t_m^0 + t_{m-1}^0 \in V((g_1^{m-1}, g_2^{m-1}, g_3^{m-1}))$ for some t_{m-1}^0. Continuing, we find that $w_0 + t_1^0 + \ldots + t_m^0 \in V(J)$ for some $t_1^0, \ldots, t_m^0$ in the other hand, if $x_0 \in V(J)$, then $x_0 \in \ker f$. Moreover, it follows from the inclusions (2.8) and Theorem 2.2 the independent of $h_1, \ldots, h_m$, so that the projection of x_0 on $W := Z_m$ belongs to the kernel of f. The corollary proved. □

We now recall some definitions from ideal theory.

Definition 2.4. The ideal $radJ$ is the radical of the ideal J, if for any positive integer k the relation $P^k \in J$ implies $P \in radJ$. If $J = radJ$, then J is a radical ideal.

Definition 2.5. An ideal J is prime if $\mathcal{P}_0(X)/J$ is integral domain, that is the algebra $\mathcal{P}_0(X)/J$ has no zero divisor ideal is maximal if $\mathcal{P}_0(X)/J$ is a field.

Theorem 2.6 (The Hilbert Nullstellensatz.). *Let J be an ideal the FT-algebra $\mathcal{P}_0(X)$, $J = (P_1, \dots, P_n)$. Then:*

(1) *If $V(J) = \varnothing$, then $J =$ (2.1).*
(2) *$I(V(J)) = radJ$.*

Proof. Since this theorem is well known for the case $\dim X < \infty$, we can assume that $\dim X = \infty$ (hence $> n$). I follows immediately from Corollary 2.3. Therefore only Point 2 requires proof.

We apply reasoning that is well known for the finite-dimensional case [12]. Let f be an arbitrary polynomial algebra $\mathcal{P}_0(X)$. Assume that $f(x) = 0 \forall x \in V(J)$. Let $y \in \mathbb{C}$ be an additional independent variable. Consider $\mathcal{P}_0(X+y)$ of polynomials on the space $X \oplus \mathbb{C}y$, that are polynomials in $\mathcal{P}_0(X)$ for each fixed $y \in \mathbb{C}$ and polynomials in $\mathbb{C}[y]$, the algebra of all polynomials in y, for each fixed $x \in X$. The algebra $\mathcal{P}_0(X+y)$ is obviously an FT-algebra. Theorem 2.2 holds in it. The polynomials $P_1, \dots, P_n$ and $fy - 1$ have no common zeros. By Point 1 of the there exist polynomials $g_1, \dots, g_{n+1} \in \mathcal{P}_0(X+y)$, such that

$$\sum_{i=1}^{n} P_i q_i + (fy-1) q_{n+1} \equiv 1,$$

and $g_1, \dots, g_{n+1}$ depend on $x \in X$ and y. Since this is an identity, it remains valid also for rational functionals the substitute $y = \frac{1}{f}$. Hence,

$$\sum P_i q_i(x, \frac{1}{f}) = 1.$$

Reducing these to a common denominator, we find that for some N

$$\sum P_i q_i'(x) f^{-N} = 1$$

or

$$\sum P_i q_i'(x) = f^N,$$

where $q_i'(x) = q_i(x, f^{-1}) f^N \in \mathcal{P}_0(X)$. But this means that f^N belongs to the ideal J. Hence $f \in radJ$ theorem is now proved. □

We now give an example of an ideal generated by an infinite number of polynomials for which the Nullstellensatz does not hold.

Example 2.7. Let H be a separable Hilbert space. Consider the ideal J generated by finite sums of polynomials $f_i(x) = (x, e_i) + a_i$, where $(\ ,\ )$ is the inner product, (e_i) is an orthonormal basis in H, and $a_i \in \mathbb{C}$. The only zero that this ideal can have is an element $\sum\limits_i a_i e_i$. But if (a_i) are chosen so that this sum diverges in H, the ideal J has no zeros. But it is obvious that the ideal J contains no units.

In the case $n = 2$ the next corollary gives a positive answer to Problem 27 of [10] (see also [13]).

Corollary 2.8. *Let $P_1, \dots, P_n$ be continuous polynomials on the Banach space X. Assume that there exists a sequence of elements $(x_i)_{i=1}^{\infty}$, $||x_i|| = 1$, such that $P_k(x_i) \to 0$ as $i \to \infty$, $1 \le k \le n$. Then the polynomials $P_1, \dots, P_n$ have a common zero.*

Proof. Suppose such is not the case. Since the algebra $\mathcal{P}_b(X)$ is an FT-algebra, according to Theorem 2.2 there exist continuous polynomials $q_1, \ldots, q_n$ such that

$$P_1 q_1 + \ldots + P_n q_n \equiv 1,$$

and this contradicts the fact that $P_k(x_i) \to \infty$, $1 \le k \le n$. The corollary is now proved. □

Now consider the topology σ on X whose closed sets are the kernels of polynomials in $\mathcal{P}_0(X)$, along with finite unions and arbitrary intersections of them. It is easy to see that this is indeed a topology. By analogy with the finite-dimensional case we call this topology the Zariski topology. We remark that for different FT-algebras we obtain different Zariski topologies. In the case of the algebra of continuous polynomials on X the Zariski topology is strictly weaker than the topology on X. In this connection the following question arises.

3. The Nullstellensatz for Algebras of Polynomials on Banach Spaces

All results of this section are proved in [14].

Let X be a Banach space, and let $\mathcal{P}(X)$ be the algebra of all continuous polynomials defined on X. Let $\mathcal{P}_0(X)$ be a subalgebra of $\mathcal{P}(X)$.

Theorem 3.1. [2] *Let Y be a complex vector space. Let A be an algebra of functions on Y such that the restriction of each $f \in A$ to any finite dimensional subspace of Y is an analytic polynomial. Let I be a proper ideal in A. Then there is a net (y_α) in Y such that $f(y_\alpha) \to 0$ for all $f \in I$.*

Here we need a technical lemma.

Lemma 3.2. [2] *Let Y be a complex vector space. Let $F = (f_1, \ldots, f_n)$ be a map from Y to $\mathbb{C}^n$ such that the restriction of each f_i to any finite dimensional space of Y is a polynomial. Then the closure of the range of F, $F(X)^-$ is an algebraic variety. Moreover there exists a finite dimensional subspace $Y_0 \subset X$ such that $F(Y_0)^- = F(X)^-$.*

Theorem 3.3. *Let $\mathcal{P}_0(X)$ be a subalgebra of $\mathcal{P}(X)$ with unity which contains all finite type polynomials. Let J be an ideal in $\mathcal{P}_0(X)$ which is generated by a finite number of polynomials $P_1, \ldots, P_n \in \mathcal{P}_0(X)$. If the polynomials $P_1, \ldots, P_n$ have no common zeros, then J is not proper.*

Proof. According to Lemma 3.2 there exists a finite dimensional subspace $Y_0 = \mathbb{C}^m \subset X$ such that $F(Y_0)^- = F(X)^-$ where $F(x) = (P_1(x), \ldots, P_n(x))$. Let $e_1, \ldots, e_m$ be a basis in Y_0 and $e_1^*, \ldots, e_m^*$ be the coordinate functionals. Denote by $P_k|_{Y_0}$ the restriction of P_k to Y_0. Since $\dim Y_0 = m < \infty$, there exists a continuous projection $T : X \to Y_0$. So any polynomial $Q \in \mathcal{P}(Y_0)$ can be exended to a polynomial $\hat{Q} \in \mathcal{P}_0(X)$ by formula $\hat{Q} = Q(T(x))$. $\hat{Q}$ belongs to $\mathcal{P}_0(X)$ becouse it is a finite type polynomial. Let us consider the map

$$G(x) = (P_1(x), \ldots, P_n(x), \hat{e}_1^*(x), \ldots, \hat{e}_m^*(x)) : X \to \mathbb{C}^{m+n}.$$

By definition of G, $G(X)^- = G(Y_0)^-$.

Suppose that J is a proper ideal in $\mathcal{P}_0(X)$ and so J is contained in a maximal ideal J_M. Let ϕ be a complex homomorphism such that $J_M = \ker \phi$. By Theorem 3.1 there exists a $\mathcal{P}_0$- convergent net (x_α) such that $\phi(P) = \lim_\alpha P(x_\alpha)$ for every $\mathcal{P}_0(X)$. Since $G(X)^- = G(Y_0)^-$, there is a net $(z_\beta \subset Y_0)$ such that $\lim_\alpha G(x_\alpha) = \lim_\beta G(z_\beta)$. Note that each polynomial $Q \in \mathcal{P}(Y_0)$ is generated by the coordinate functionals. Thus $\lim_\beta Q(z_\beta) = \lim_\alpha \hat{Q}(x_\alpha) = \phi(Q)$. Also $\lim_\beta P_k|_{Y_0}(z_\beta) = \lim_\alpha P_k(x_\alpha) = \phi(P_k)$, $k = 1, \ldots, n$. On the other hand, every $\mathcal{P}_0$-convergent net on a finite dimensional subspace is weakly convergent and so it converges to a point $x_0 \in Y_0 \subset X$. Thus $P_k(x_0) = 0$ for $1 \le k \le n$ that contradicts the assumption that $P_1, \dots, P_n$ have no common zeros. □

A subalgebra A_0 of an algebra A is called *factorial* if for every $f \in A_0$ the equality $f = f_1 f_2$ implies that $f_1 \in A_0$ and $f_2 \in A_0$.

Theorem 3.4 (Hilbert Nullstellensatz Theorem). *Let $\mathcal{P}_0(X)$ be a factorial subalgebra in $\mathcal{P}(X)$ which contains all polynomials of finite type and let J be an ideal of $\mathcal{P}_0(X)$ which is generated by a finite sequence of polynomials $P_1, \dots, P_n$. Then $radJ \subset \mathcal{P}_0(X)$ and*

$$I[V(J)] = radJ$$

in $\mathcal{P}_0(X)$.

Proof. Since $\mathcal{P}_0(X)$ is factorial, $radJ \subset \mathcal{P}_0(X)$ for every ideal $J \in \mathcal{P}_0(X)$. Evidently, $I[V(J)] \supset radJ$. Let $P \in \mathcal{P}_0(X)$ and $P(x) = 0$ for every $x \in V(J)$. Let $y \in \mathbb{C}$ be an additional "independent variable" which is associated with a basis vector e of an extra dimension. Consider a Banach space $X \oplus \mathbb{C}e = \{x + ye : x \in X, y \in \mathbb{C}\}$. We denote by $\mathcal{P}_0(X) \otimes \mathcal{P}(\mathbb{C})$ the algebra of polynomials on $X \oplus \mathbb{C}e$ such that every polynomial in $\mathcal{P}_0(X) \otimes \mathcal{P}(\mathbb{C})$ belongs to $\mathcal{P}_0(X)$ for arbitrary $y \in \mathbb{C}$. The polynomials $P_1, \dots, P_n, Py - 1$ have no common zeros. By Theorem 3.3 there are polynomials $Q_1, \dots, Q_{n+1} \in \mathcal{P}_0(X) \otimes \mathcal{P}(\mathbb{C})$ such that

$$\sum_{i=1}^{n} P_i Q_i + (Py - 1) Q_{n+1} \equiv 1.$$

Since it is an identity it will be still true for all vectors x such that $P(x) \neq 0$ if we substitute $y = 1/P(x)$. Thus

$$\sum_{i=1}^{n} P_i(x) Q_i(x, 1/P(x)) = 1.$$

Taking a common denominator, we find that for some positive integer N,

$$\sum_{i=1}^{n} P_i(x) Q_i'(x) P^{-N}(x) = 1$$

or

$$\sum_{i=1}^{n} P_i(x) Q_i'(x) = P^N(x), \tag{3.1}$$

where $Q'(x) = Q(x, P^{-1}) P^N(x) \in \mathcal{P}_0(X)$. The equality (3.1) holds on an open subset $X \ \ker P$, so it holds for every $x \in X$. But it means that P^N belongs to J. So $P \in radJ$. □

4. The Nullstellensatz for Algebras of Symmetric Polynomials on ℓ_p

Let X be a Banach space, and let $\mathcal{P}(X)$ be the algebra of all continuous polynomials defined on X. Let $\mathcal{P}_0(X)$ be a subalgebra of $\mathcal{P}(X)$. A sequence $(G_i)_i$ of polynomials is called an *algebraic basis* of $\mathcal{P}_0(X)$ if for every $P \in \mathcal{P}_0(X)$ there is $q \in \mathcal{P}(\mathbb{C})$ for some n such that $P(x) = q(G_1(x), \dots, G_n(x))$; in other words, if G is the mapping $x \in X \rightsquigarrow G(x) := (G_1(x), \dots, G_n(x)) \in \mathbb{C}^n$, then $P = q \circ G$.

Let $\mathcal{P}_s(X)$ be the algebra of all symmetric polynomials. Let $\langle p\rangle$ be the smallest integer that is greater than or equal to p. In [5], it is proved that the polynomials $F_k(\sum a_i e_i) = \sum a_i^k$ for $k = \langle p\rangle, \langle p\rangle + 1, \ldots$ form an algebraic basis in $\mathcal{P}_s(\ell_p)$. So there are no symmetric polynomials of degree less than $\langle p\rangle$ in $\mathcal{P}_s(\ell_p)$ and if $\langle p_1\rangle = \langle p_2\rangle$, then $\mathcal{P}_s(\ell_{p_1}) = \mathcal{P}_s(\ell_{p_2})$. Thus, without loss of generality we can consider $\mathcal{P}_s(\ell_p)$ only for integer values of p. Throughout, we shall assume that p is an integer, $1 \le p < \infty$.

It is well known [8] that for $n < \infty$ any polynomial in $\mathcal{P}_s(\mathbb{C}^n)$ is uniquely representable as a polynomial in the *elementary symmetric polynomials* $(R_i)_{i=1}^n$, $R_i(x) = \sum_{k_1<\ldots<k_i} x_{k_1} \ldots x_{k_i}$.

In paper [1] was proof next results.

Lemma 4.1. *Let $\{G_1, \ldots, G_n\}$ be an algebraic basis of $\mathcal{P}_s(\mathbb{C}^n)$. For any $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$, there is $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ such that $G_i(x) = \xi_i$, $i = 1, \ldots, n$. If for some $y = (y_1, \ldots, y_n)$, $G_i(y) = \xi_i$ $i = 1, \ldots, n$, then $x = y$ up to a permutation.*

Proof. First, we suppose that $G_i = R_i$. Then, according to the Vieta formulae [8], the solutions of the equation

$$x^n - \xi_1 x^{n-1} + \ldots + (-1)^n \xi_n = 0$$

satisfy the conditions $R_i(x) = \xi_i$, and so $x = (x_1, \ldots, x_n)$ as required. Now let G_i be an arbitrary algebraic basis of $\mathcal{P}_s(\mathbb{C}^n)$. Then $R_i(x) = v_i(G_1(x), \ldots, G_n(x))$ for some polynomials v_i on $\mathbb{C}^n$. Setting v as the polynomial mapping $x \in \mathbb{C}^n \rightsquigarrow v(x) := (v_1(x), \ldots, v_n(x)) \in \mathbb{C}^n$, we have $R = v \circ G$.

As the elementary symmetric polynomials also form a basis, there is a polynomial mapping $w : \mathbb{C}^n \to \mathbb{C}^n$ such that $G = w \circ R$; hence $R = (v \circ w) \circ R$ so $v \circ w = \mathrm{id}$. Then v and w are inverse to each other, since $w \circ v$ coincides with the identity on the open set, $\mathrm{Im}(w)$. In particular, v is one-to-one.

Now, the solutions $x_1, \ldots, x_n$ of the equation

$$x^n - v_1(\xi_1, \ldots, \xi_n) x^{n-1} + \ldots + (-1)^n v_n(\xi_1, \ldots, \xi_n) = 0$$

satisfy the conditions $R_i(x) = v_i(\xi)$, $i = 1, \ldots, n$. That is, $v(\xi) = R(x) = v(G(x))$, and hence $\xi = G(x)$. $\square$

Corollary 4.2. *Given $(\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$, there is $x \in \ell_p^{n+p-1}$ such that*

$$F_p(x) = \xi_1, \ldots F_{p+n-1}(x) = \xi_n.$$

Proposition 4.3 (Nullstellensatz). *Let $P_1, \ldots, P_m \in \mathcal{P}_s(\ell_p)$ be such that* $\ker P_1 \cap \ldots \cap \ker P_m = \varnothing$. *Then there are $Q_1, \ldots, Q_m \in \mathcal{P}_s(\ell_p)$ such that $\sum\limits_{i=1}^m P_i Q_i \equiv 1$.*

Proof. Let $n = \max\limits_i(\deg P_i)$. We may assume that $P_i(x) = q_i(F_p(x), \ldots, F_n(x))$ for some $g_i \in \mathcal{P}(\mathbb{C}^{n-p+1})$. Let us suppose that at some point $\xi \in \mathbb{C}^{n-p+1}$, $\xi = (\xi_1, \ldots, \xi_{n-p+1})$ and $g_i(\xi) = 0$. Then by Corollary 4.2 there is $x_0 \in \ell_p$ such that $F_i(x_0) = \xi_i$. So the common set of zeros of all q_i is empty. Thus by the Hilbert Nullstellensatz there are polynomials $q_1, \ldots, q_m$ such that $\sum\limits_i g_i q_i \equiv 1$. Put $Q_i(x) = q_i(F_p(x), \ldots, F_n(x))$. $\square$

5. The Nullstellensatz for Algebras of Block-Symmetric Polynomials

Let

$$\mathcal{X}^2 = \oplus_{\ell_1} \mathbb{C}^2$$

be an infinite ℓ_1-sum of copies of Banach space $\mathbb{C}^2$. So any element $\overline{x} \in \mathcal{X}^2$ can be represented as a sequence $\overline{x} = (x_1, \ldots, x_n, \ldots)$, where $x_n \in \mathbb{C}^2$, with the norm $\|\overline{x}\| = \sum\limits_{k=1}^{\infty} \|x_k\|$.

A polynomial P on the space $\mathcal{X}^2$ is called *block-symmetric* (or *vector-symmetric*) if:

$$P\left(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, \ldots\right) = P\left(x_1, \ldots, x_n, \ldots\right),$$

where $x_i \in \mathbb{C}^2$ for every permutation σ on the set $\mathbb{N}$. Let us denote by $\mathcal{P}_{vs}(\mathcal{X}^2)$ the algebra of block-symmetric polynomials on $\mathcal{X}^2$.

In paper [7] it was shown that the algebraic basis of algebra $\mathcal{P}_{vs}(\mathcal{X}^2)$ is form by polynomials

$$H^{p,n-p}(x,y) = \sum_{i=1}^{\infty} x_i^p y_i^{n-p},$$

where $0 \le p \le n$, $(x_i, y_i) \in \mathbb{C}^2$.

Let us denote by $\mathcal{P}_{vs}^m(\mathcal{X}^2)$ the subalgebra of $\mathcal{P}_{vs}(\mathcal{X}^2)$ which is generated by polynomials

$$H^{1,0,}(x,y), \ldots, H^{p,n-p}(x,y).$$

The number of these elements is equal to m and we denote by τ_{vs}^m the system of generators of algebra $\mathcal{P}_{vs}^m(\mathcal{X}^2)$.

Let $(x,y), (z,t) \in \mathcal{X}^2$,

$$(x,y) = \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \ldots, \begin{pmatrix} x_m \\ y_m \end{pmatrix}, \ldots \right)$$

and

$$(z,t) = \left(\begin{pmatrix} z_1 \\ t_1 \end{pmatrix}, \ldots, \begin{pmatrix} z_m \\ t_m \end{pmatrix}, \ldots \right)$$

where $(x_i, y_i), (z_i, t_i) \in \mathbb{C}^2$. We put

$$(x,y) \bullet (z,t) = \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} z_1 \\ t_1 \end{pmatrix}, \ldots, \begin{pmatrix} x_m \\ y_m \end{pmatrix}, \begin{pmatrix} z_m \\ t_m \end{pmatrix}, \ldots \right)$$

and define

$$\mathcal{T}_{(z,t)}(f)(x,y) := f((x,y) \bullet (z,t)). \tag{5.1}$$

We will say that $(x,y) \to (x,y) \bullet (z,t)$ is the block symmetric translation and the operator $\mathcal{T}_{(z,t)}$ is the symmetric translation operator. Evidently, we have that

$$H^{k_1,k_2}((x,y) \bullet (z,t)) = H^{k_1,k_2}(x,y) + H^{k_1,k_2}(z,t)$$

for all k_1, k_2.

For some positive number k denote by $\alpha_{0,k}, \alpha_{1,k}, \ldots, \alpha_{k-1,k}$ complex k^{th} roots of the unity, namely $\alpha_{m,k} = e^{2mi\pi/k}$. The following lemma is well known.

Lemma 5.1. *For some pos* $\sum\limits_{m=0}^{k-1} \alpha^n_{m,k} = h \Big\{$ *umber n*

$$\sum_{m=0}^{k-1} \alpha^n_{m,k} = h \begin{cases} k & \text{if} \quad n = 0 \mod k \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 5.2. *For any $H^{p,n-p} \in \tau^m_{vs}$ on $\mathcal{X}^2$ and for any $\xi_{p,n-p}$ there exist a vector*

$$(x,y)_{p,n-p} = \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \ldots, \begin{pmatrix} x_{N_{p,n-p}} \\ y_{N_{p,n-p}} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ldots \right)$$

in $\mathcal{X}^2$ such that $H^{p,n-p} = \xi_{p,n-p}$, $H^{l_1,l_2} = 0$ for all $l_1 \neq p, l_2 \neq n-p$.

Proof. Let us consider two cases:

(1) $p = 0$ or $n = p$;
(2) $p \neq 0, n \neq p$.

1. If $p = 0$ or $n = p$, then the polynomials $H^{0,n}(x,y) = F_p(y)$ and $H^{p,0}(x,y) = F_p(x)$ are symmetric relatively vectors $y = (y_1, \ldots, y_n, \ldots)$, $x = (x_1, \ldots, x_n, \ldots)$ respectively. In the paper [1, p. 57] is proof that for symmetric polynomial $F_k(x) = \sum\limits_{i=1}^{\infty} x_i^k$ exist the vector $x_0 = (x_0^1, x_0^2, \ldots, x_0^n, \ldots) \in \ell_1$ such that $F_k(x_0) = \xi_{k0}$, $F_j(x_0) = 0$. Then for the polynomial $H^{p,0}(x,y)$ there exists vector $(x_0, 0)_{p,0}$ such that $H^{p,0}((x_0,0)_{p,0}) = \xi_{p,0}$ and $H^{l_1,l_2}((x_0,0)_{p,0}) = 0$ for all $l_1 \neq p, l_2 \neq 0$. If we have $p = 0$ then there exists vector $(0, y_0)_{0,n}$ such that $H^{0,n}((0,y_0)_{0,n}) = \xi_{0,n}$ and $H^{l_1,l_2}((0,y_0)_{0,n}) = 0$ for all $l_1 \neq 0, l_2 \neq n$.

2. For the second case we consider polynomials

$$H^{p,k-p}(x,y) = \sum_{i=1}^{\infty} x_i^p y_i^{k-p} \in \tau^m_{vs}$$

of degree k, where $1 \leq p < k$. First we assume that $p \geq k - p$, $p \geq \frac{k}{2}$ and consider the vector

$$(\overline{x}, \overline{y}) = \left(\begin{pmatrix} a\big(\alpha_{0,p(n+1)}\big)^{n+1-(k-p)} \\ b\big(\alpha_{0,p(n+1)}\big)^{p} \end{pmatrix}, \begin{pmatrix} a\big(\alpha_{1,p(n+1)}\big)^{n+1-(k-p)} \\ b\big(\alpha_{1,p(n+1)}\big)^{p} \end{pmatrix}, \ldots, \right.$$

$$\left. \begin{pmatrix} a\big(\alpha_{p(n+1)-1,p(n+1)}\big)^{n+1-(k-p)} \\ b\big(\alpha_{p(n+1)-1,p(n+1)}\big)^{p} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ldots \right),$$

where $\alpha_{i,p(n+1)}$ is the i^{th} roots of complex $p(n+1)$ roots of the unity.

According to Lemma 5.1 we have $H^{p,k-p}(\overline{x},\overline{y}) = p(n+1)a^p b^{k-p}$. On the system of generating τ^m_{vs} there exists a polynomial which is not equal to zero at $(\overline{x},\overline{y})$. Let us denote by $H^{p_1,k_1-p_1}, \ldots, H^{p_l,k_l-p_l}$, $k_l \leq n$ the polynomials such that

$$p_i(n+1-(k-p)) + p(k_i - p_i) = 0 \mod p(n+1), \qquad i = 1, \ldots, l.$$

For this polynomials we have $H^{p_i,k_i-p_i}(\overline{x},\overline{y}) = p_i(n+1)a^{p_i}b^{k_i-p_i}$, $i = 1, \ldots, l$. All other polynomials of the system τ^m_{vs} are equal of zero at $(\overline{x},\overline{y})$.

We note that for all $i = 1, \ldots, l$ $k_i \neq k$. Indeed let $k_1 = k$. In the case $p_1 < p$ we obtain

$$p_1(n+1-(k-p)) + p(k_1 - p_1) = (n+1)p_1 + k(p - p_1) = (n+1-k)p_1 + kp < p(n+1).$$

From this inequality it follows

$$p_1(n+1-(k-p)) + p(k_1 - p_1) \neq 0 \mod p(n+1),$$

that contradicts above hypothesis.

In the case $p_1 > p$ we obtain

$$p_1(n+1-(k-p)) + p(k_1 - p_1) = (n+1)p_1 + k(p - p_1) = (n+1-k)p_1 + kp < p_1(n+1).$$

From this inequality it follows that for the condition $p_1(n+1) = 0 \mod p(n+1)$ necessary $p_1 = sp, s > 1, s \in \mathbb{N}$.

Since $p > \frac{k}{2}$, then $p_1 > s\frac{k}{2}$. Since $s > 1$ and $s \in \mathbb{N}$, then if $s_{\min} = 2$ we obtain that $p_1 > k$, wich is impossible. Therefore, $k_i \neq k$.

Now we show that $k < k_i$ for all $i = 1, \dots, l$. Indeed let $i = 1$ $k_1 < k$. For the polynomial $H^{p_1, k_1 - p_1}$ we have $p_1(n+1-(k-p)) + p(k_1 - p_1) = 0 \mod p(n+1)$. From inequality $k_1 < k$ it follows that

$$p_1(n+1-(k-p)) + p(k_1 - p_1) = p_1(n+1-k) + pk. \tag{5.2}$$

If $p_1 < p$ we obtain:

$$p_1(n+1-k) + pk < p(n+1).$$

Therefore $p_1(n+1-(k-p)) + p(k_1 - p_1) \neq 0 \mod p(n+1)$.

If $p_1 \geq p$, then

$$p_1(n+1-k) + pk \leq p_1(n+1).$$

In order to last expression of inequality will evenly divided on $p(n+1)$ necessary that $p_1 = sp$. Since $p > \frac{k}{2}$, then $p > \frac{k_1}{2}$, then $p_1 > s\frac{k_1}{2}$. If $s = 1$ we obtain that $p_1(n+1-(k-p)) + p(k_1 - p_1) = p(n+1-(k-p)) + p(k_1 - p) = p(n+1) - p(k - k_1) < p(n+1)$. Therefore on this case $p_1(n+1-(k-p)) + p(k_1 - p_1) \neq 0 \mod p(n+1)$. If $s \geq 2$ we obtain $p_1 > k_1$, wich is impossible. Therefore $k < k_i$ for all $i = 1, \dots, l$.

We will show that $p_i = sp$, $k_i = sk$ for all $i = 1, \dots, l$. Indeed from

$$p(n+1-(k-p)) + p(k-p) = 0 \mod p(n+1)$$

it follows that

$$mp(n+1-(k-p)) + pm(k-p) = 0 \mod p(n+1),$$

where $m > 1$ (the case $m < 1$ is impossible because $mk < k$). Therefore we obtain the polynomials $H^{mp, m(k-p)}$, which will be among the polynomials $H^{p_1, k_1 - p_1}, \dots, H^{p_l, k_l - p_l}$. We suppose that there exist polynomials $H^{p+s_1, k-p+s_2}$, where $s_1 < p$, $s_2 < k - p$.

Then

$$\begin{aligned}(p+s_1)(n+1-(k-p)) + p(k-p+s_2) &= p(n+1-(k-p)) + p(k-p) \\ &+ s_1(n+1-(k-p))ps_2.\end{aligned}$$

Since $p(n+1-(k-p)) + p(k-p) = 0 \mod p(n+1)$, then should performed the codition

$$s_1(n+1-(k-p)) + ps_2 = 0 \mod p(n+1),$$

wich is impossible because $s_1(n+1-(k-p)) + ps_2 < p(n+1-(k-p)) + p(k-p) = p(n+1)$. Therefore all polynomials $H^{p_1, k_1 - p_1}, \dots, H^{p_l, k_l - p_l}$ are of the form $H^{mp, m(k-p)}$, $m = 2, \dots, w$ where $wk < n+1$. Therefore the polynomials $H^{p_1, k_1 - p_1}, \dots, H^{p_l, k_l - p_l}$ we can mark as

$$H^{p_1, k_1 - q_1} = H^{2p, 2(k-p)}, H^{p_2, k_2 - p_2} = H^{3p, 3(k-p)}, \dots, H^{p_l, k_l - p_l} = H^{(l+1)p, (l+1)(k-p)},$$

where $(l+1)k < n+1$.

Next we concider the vector

$$(\overline{\overline{x}},\overline{\overline{y}}) = \left(\begin{pmatrix} a\sqrt[2k]{-1}\left(\alpha_{0,p(n+1)}\right)^{n+1-(k-p)} \\ b\sqrt[2k]{-1}\left(\alpha_{0,p(n+1)}\right)^{p} \end{pmatrix}, \ldots, \begin{pmatrix} a\sqrt[2k]{-1}\left(\alpha_{p(n+1)-1,p(n+1)}\right)^{n+1-(k-p)} \\ b\sqrt[2k]{-1}\left(\alpha_{p(n+1)-1,p(n+1)}\right)^{p} \end{pmatrix}, \ldots, \right.$$

$$\begin{pmatrix} a\sqrt[(l+1)k]{-1}\left(\alpha_{0,p(n+1)}\right)^{n+1-(k-p)} \\ b\sqrt[(l+1)k]{-1}\left(\alpha_{0,p(n+1)}\right)^{p} \end{pmatrix}, \ldots, \begin{pmatrix} a\sqrt[(l+1)k]{-1}\left(\alpha_{p(n+1)-1,p(n+1)}\right)^{n+1-(k-p)} \\ b\sqrt[(l+1)k]{-1}\left(\alpha_{p(n+1)-1,p(n+1)}\right)^{p} \end{pmatrix},$$

$$\begin{pmatrix} a\sqrt[2k]{-1}\sqrt[3k]{-1}\left(\alpha_{0,p(n+1)}\right)^{n+1-(k-p)} \\ b\sqrt[2k]{-1}\sqrt[3k]{-1}\left(\alpha_{0,p(n+1)}\right)^{p} \end{pmatrix}, \ldots, \begin{pmatrix} a\sqrt[2k]{-1}\sqrt[3k]{-1}\left(\alpha_{p(n+1)-1,p(n+1)}\right)^{n+1-(k-p)} \\ b\sqrt[2k]{-1}\sqrt[3k]{-1}\left(\alpha_{p(n+1)-1,p(n+1)}\right)^{p} \end{pmatrix}, \ldots,$$

$$\begin{pmatrix} a\sqrt[(i+1)k]{-1}\sqrt[(j+1)k]{-1}\left(\alpha_{0,p(n+1)}\right)^{n+1-(k-p)} \\ b\sqrt[(i+1)k]{-1}\sqrt[(j+1)k]{-1}\left(\alpha_{0,p(n+1)}\right)^{p} \end{pmatrix}, \ldots,$$

$$\begin{pmatrix} a\sqrt[(i+1)k]{-1}\sqrt[(j+1)k]{-1}\left(\alpha_{p(n+1)-1,p(n+1)}\right)^{n+1-(k-p)} \\ b\sqrt[(i+1)k]{-1}\sqrt[k(j+1)]{-1}\left(\alpha_{p(n+1)-1,p(n+1)}\right)^{p} \end{pmatrix}, \ldots,$$

$$\begin{pmatrix} a\sqrt[lk]{-1}\sqrt[(l+1)k]{-1}\left(\alpha_{0,p(n+1)}\right)^{n+1-(k-p)} \\ b\sqrt[lk]{-1}\sqrt[(l+1)k]{-1}\left(\alpha_{0,p(n+1)}\right)^{p} \end{pmatrix}, \ldots,$$

$$\begin{pmatrix} a\sqrt[lk]{-1}\sqrt[(l+1)k]{-1}\left(\alpha_{p(n+1)-1,p(n+1)}\right)^{n+1-(k-p)} \\ b\sqrt[lk]{-1}\sqrt[(l+1)k]{-1}\left(\alpha_{p(n+1)-1,p(n+1)}\right)^{p} \end{pmatrix}, \ldots,$$

$$\begin{pmatrix} a\sqrt[i_2k]{-1}\ldots\sqrt[i_lk]{-1}\left(\alpha_{0,p(n+1)}\right)^{n+1-(k-p)} \\ b\sqrt[i_1k]{-1}\ldots\sqrt[i_{l-1}k]{-1}\left(\alpha_{0,p(n+1)}\right)^{p} \end{pmatrix}, \ldots,$$

$$\begin{pmatrix} a\sqrt[i_1k]{-1}\ldots\sqrt[i_{l-1}k]{-1}\left(\alpha_{p(n+1)-1,p(n+1)}\right)^{n+1-(k-p)} \\ b\sqrt[i_1k]{-1}\ldots\sqrt[i_{l-1}k]{-1}\left(\alpha_{p(n+1)-1,p(n+1)}\right)^{p} \end{pmatrix}, \ldots,$$

$$\begin{pmatrix} a\sqrt[2k]{-1}\ldots\sqrt[lk]{-1}\sqrt[(l+1)k]{-1}\left(\alpha_{0,p(n+1)}\right)^{n+1-(k-p)} \\ b\sqrt[2k]{-1}\ldots\sqrt[lk]{-1}\sqrt[(l+1)k]{-1}\left(\alpha_{0,p(n+1)}\right)^{p} \end{pmatrix}, \ldots,$$

$$\left. \begin{pmatrix} a\sqrt[2k]{-1}\ldots\sqrt[lk]{-1}\sqrt[(l+1)k]{-1}\left(\alpha_{p(n+1)-1,p(n+1)}\right)^{n+1-(k-p)} \\ b\sqrt[2k]{-1}\ldots\sqrt[lk]{-1}\sqrt[(l+1)k]{-1}\left(\alpha_{p(n+1)-1,p(n+1)}\right)^{p} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ldots \right).$$

Then we obtain that

$$\begin{aligned} H^{ip,i(k-p)}((\overline{x},\overline{y}) \bullet (\overline{\overline{x}},\overline{\overline{y}})) &= a^{ip}b^{i(k-p)}\Bigg(p(n+1) - p(n+1) + p(n+1)\sum_{\substack{j=1\\ j\neq i}}^{l}\left(\sqrt[jk]{-1}\right)^{ik} \\ &- p(n+1)\sum_{\substack{j=1\\ j\neq i}}^{l}\left(\sqrt[jk]{-1}\right)^{ik} + \dots \\ &+ p(n+1)\sum_{\substack{j_1<\dots<j_{l-1}\\ j_m\neq i}}^{l}\left(\sqrt[j_1k]{-1}\dots\sqrt[j_{l-1}k]{-1}\right)^{ik} \\ &- p(n+1)\sum_{\substack{j_1<\dots<j_{l-1}\\ j_m\neq i}}^{l}\left(\sqrt[j_1k]{-1}\dots\sqrt[j_{l-1}k]{-1}\right)^{ik}\Bigg) = 0. \end{aligned}$$

For $H^{p,k-p}$ we obtain

$$\begin{aligned} H^{p,k-p}((\overline{x},\overline{y}) \bullet (\overline{\overline{x}},\overline{\overline{y}})) &= p(n+1)a^{p}b^{k-p}\Bigg(1 + \sum_{j=1}^{l}\sqrt[j+1]{-1} + \dots \\ &+ \sum_{j_1<\dots<j_{l-1}}^{l}\sqrt[j_1]{-1}\dots\sqrt[j_{l-1}]{-1} + \sqrt[2]{-1}\dots\sqrt[l]{-1}\sqrt[l+1]{-1}\Bigg). \end{aligned} \tag{5.3}$$

We denote by M the next condition

$$M = 1 + \sum_{j=1}^{l}\sqrt[j+1]{-1} + \dots + \sum_{j_1<\dots<j_{l-1}}^{l}\sqrt[j_1]{-1}\dots\sqrt[j_{l-1}]{-1} + \sqrt[2]{-1}\dots\sqrt[l]{-1}\sqrt[l+1]{-1}.$$

If we choice $(j+1)^{th}$ a complex root of -1, $j = 1,\dots ł$ such that $M \neq 0$ to zero, we obtain $H^{p,k-p}((\overline{x},\overline{y}) \bullet (\overline{\overline{x}},\overline{\overline{y}})) \neq 0$.

If we substitute to (5.3)

$$a = \frac{1}{\sqrt[p]{(k-p)(n+1)M}}\sqrt[p]{\xi_{p,k-p}}, \qquad b = 1$$

we obtain

$$H^{p,k-p}((\overline{x},\overline{y}) \bullet (\overline{\overline{x}},\overline{\overline{y}})) = H^{p,k-p}((x,y)_{p,k-p}) = \xi_{p,k-p}.$$

In the case $p < k-p$ we consider the vector

$$\begin{aligned} (\overline{x},\overline{y}) = \Bigg(&\begin{pmatrix} a\left(\alpha_{0,(k-p)(n+1)}\right)^{k-p} \\ b\left(\alpha_{0,(k-p)(n+1)}\right)^{n+1-p} \end{pmatrix}, \begin{pmatrix} a\left(\alpha_{1,(k-p)(n+1)}\right)^{k-p} \\ b\left(\alpha_{1,(k-p)(n+1)}\right)^{n+1-p} \end{pmatrix}, \dots, \\ &\begin{pmatrix} a\left(\alpha_{(k-p)(n+1)-1,(k-p)(n+1)}\right)^{k-p} \\ b\left(\alpha_{(k-p)(n+1)-1,(k-p)(n+1)}\right)^{n+1-p} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \dots\Bigg), \end{aligned}$$

where $\alpha_{i,(k-p)(n+1)}$ is i^{th} root of $(k-p)(n+1)$ complex root of the unity. For this case the proof is the same like in the case $p \geq k-p$. $\square$

Corollary 5.3. *Let* $\tau_{vs}^m = \{H^{\tilde{p},j-\tilde{p}}(x,y), 0 \leq \tilde{p} \leq j, j = 1,\dots,n\}$, $j \leq m$. *Then for each* $\xi = (\xi_{1,0},\dots,\xi_{p,k-p},\dots,\xi_{p',k'-p'}) \in \mathbb{C}^m$ *there is* $(x,y)_{p,k-p} \in \mathcal{X}^2$ *such that* $H^{p,k-p}((x,y)_{pq}) = \xi_{p,k-p}$.

Proposition 5.4. *Let* $P_1,\dots,P_m \in \mathcal{P}_{vs}(\mathcal{X}^2)$ *such that* $\ker P_1 \cap \dots \cap \ker P_m = \varnothing$. *Then there are* $Q_1,\dots,Q_m \in \mathcal{P}_{vs}(\mathcal{X}^2)$ *such that*

$$\sum_{i=1}^{m} P_i Q_i = 1.$$

Proof. For the proof we use the same method as in [1, p. 58]. Let $n = \max_i(\deg P_i)$. We may assume that $P_i(x,y) = q_i(H^{1,0},\dots,H^{l_1,k-l_1})$ for some $q_i \in \mathcal{P}(\mathbb{C}^n)$, where $0 \leq l_1 \leq k$, n is number of polynomials $H^{l_1,k-l_1}$. Let us suppose that at some point $\xi \in \mathbb{C}^n$, $\xi = (\xi_{1,0},\dots,\xi_{p,k-p},\dots,\xi_{p',k'-p'})$ and $g_i(\xi) = 0$. Then by Corollary 5.3 there is $(x,y)_{p,k-p} \in \mathcal{X}^2$ such that $H^{p,k-p}((x,y)_{p,k-p}) = \xi_{p,k-p}$. So the common set of zeros of all q_i is empty. Thus by the Hilbert Nullstellensatz there are polynomials $g_1,\dots,g_m$ such that $\sum_i q_i g_i \equiv 1$. Put $Q_i(x,y) = g_i(H^{1,0},\dots,H^{l_1,k-l_1})$. □

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У роботі доведено теореми Гільберта про нулі для поліномів на нескінченно вимірному комплексному просторі, для симетричних та блочно-симетричних поліномів.

Ключові слова: поліноми, симетричні поліноми, блочно-симетричні поліноми, алгебра поліномів, теорема Гільберта про нулі, алгебраїчний базис.

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HYPERCYCLIC COMPOSITION OPERATORS

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Abstract. In this paper we give survey of hypercyclic composition operators. In pacticular, we represent new classes of hypercyclic composition operators on the spaces of analytic functions.
Keywords: hypercyclic operators, functional spaces, polynomial automorphisms, symmetric functions

1. INTRODUCTION

Hypercyclicity is a young and rapidly evolving branch of functional analysis, which was probably born in 1982 with the Ph.D. thesis of Kitai [24]. It has become rather popular, thanks to the efforts of many mathematicians. In particular, the seminal paper [17] by Godefroy and Shapiro, the survey [20] by Grosse-Erdmann and useful notes [37] by Shapiro have had a considerable influence on both its internal development. Let us recall a definition of hypercyclic operator.

Definition 1.1. Let X be a Fréchet linear space. A continuous linear operator $T : X \to X$ is called *hypercyclic* if there is a vector $x_0 \in X$ for which the orbit under T, $\mathrm{Orb}(T, x_0) = \{x_0, Tx_0, T^2x_0, \ldots\}$ is dense in X. Every such vector x_0 is called a *hypercyclic vector* of T.

The investigation of hypercyclic operators has relation to invariant subspaces problem. It is easy to check that if every nonzero vector of X is hypercyclic for T, then T has no closed invariant subsets, and so no closed invariant subspaces as well. In his paper [32] Read shows that there exists continuous linear operator on ℓ_1 for which every nonzero vector is hypercyclic. It is still open problem does exist a linear continuous operator on a separable Hilbert space without closed invariant subspaces.

The classical Birkhoff's theorem [7] asserts that any operator of composition with translation $x \mapsto x + a$, $T_a \colon f(x) \mapsto f(x + a)$, $(a \neq 0)$ is hypercyclic on the space of entire functions $H(\mathbb{C})$ on the complex plane $\mathbb{C}$, endowed with the topology of uniform convergence on compact subsets. The Birkhoff's translation T_a has also been regarded as a differentiation operator

$$T_a(f) = \sum_{n=0}^{\infty} \frac{a^n}{n!} D^n f.$$

In 1941, Seidel and Walsh [35] obtained an analogue for non-Euclidean translates in the unit disk $\mathbb{D}$. Variants and strengthenings of the theorems of Birkhoff and Seidel and Walsh were found by Heins [22], Luh [25], [26] and Shapiro [36], while Gauthier [15] gave a new proof of Birkhoff's theorem.

In 1952, MacLane [27] showed that there exists an entire function f whose derivatives $f^{(n)}$ ($n \in \mathbb{N}_0$) form a dense set in the space $H(\mathbb{C})$ of entire functions, in other words, that the differentiation operator D is hypercyclic on $H(\mathbb{C})$. This result was rederived by Blair and Rubel [8]. Duyos-Ruiz [14] showed the residuality of set of entire functions that are hypercyclic for D; see also [16] and [19].

The most remarkable generalization of MacLane's theorem, which at the same time also includes Birkhoff's theorem was proved by Godefroy and Shapiro in [17]. They showed that if $\varphi(z) = \sum\limits_{|\alpha|\geq 0} c_\alpha z^\alpha$ is a non-constant entire function of exponential type on $\mathbb{C}^n$, then the operator

$$f \mapsto \sum_{|\alpha|\geq 0} c_\alpha D^\alpha f, \qquad f \in H(\mathbb{C}^n) \tag{1.1}$$

is hypercyclic.

We fix $n \in \mathbb{N}$ and denote by $T_a : \mathbb{C}^n \to \mathbb{C}^n$ the translation operator $T_a f(z) = f(z+a)$ for $a \in \mathbb{C}^n$ and by $D_k : \mathbb{C}^n \to \mathbb{C}^n$ the differentiation operator $D_k f(z) = \frac{\partial f}{\partial z_k}(z)$ for $1 \leq k \leq n$.

Theorem 1.2. *(Godefroy, Shapiro). Let T be a continuous linear operator on $H(\mathbb{C}^n)$ that commutes with all translation operators T_a, $a \in \mathbb{C}^n$ (or, equivalently, with all differentiation operators D_k, $1 \leq k \leq n$). If T is not a scalar multiple of the identity, then T is hypercyclic.*

Further hypercyclicity for differential and related operators are obtained by Mathew [28], Bernal [4] for spaces $H(O)$, $O \subset \mathbb{C}$ open; by Bonet [9] for weighted inductive limits of spaces of holomorphic functions.

Let us recall that an operator C_φ on $H(\mathbb{C}^n)$ is said to be a *composition operator* if $C_\varphi f(x) = f(\varphi(x))$ for some analytic map $\varphi\colon \mathbb{C}^n \to \mathbb{C}^n$. It is known that only translation operator T_a for some $a \neq 0$ is a hypercyclic composition operator on $H(\mathbb{C})$ [6]. However, if $n > 1$, $H(\mathbb{C}^n)$ supports more hypercyclic composition operators. In [5] Bernal-González established some necessary and sufficient conditions for a composition operator by an affine map to be hypercyclic. In particular, in [5] it is proved that a given affine automorphism $S = A + b$ on $\mathbb{C}^n$, the composition operator $C_S : f(x) \mapsto f(S(x))$ is hypercyclic if and only if the linear operator A is bijective and the vector b is not in the range of $A - I$.

In [11] Chan and Shapiro show that T_a is hypercyclic in various Hilbert spaces of entire functions on $\mathbb{C}$. More detailed, they considered Hilbert spaces of entire functions of one complex variable $f(z) = \sum\limits_{n=1}^{\infty} f_n z^n$ with norms $\|f\|^2_{2,\gamma} = \sum\limits_{n=1}^{\infty} \gamma_n^{-2} |f_n|^2$ for appropriated sequence of positive numbers and shown that if $n\gamma_n/\gamma_{n-1}$ is monotonically decreasing, then T_a is hypercyclic.

In [34] Rolewicz proved that even though the backward shift operator $B : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ on the space of square summable sequences defined by

$$B(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots)$$

is not hypercyclic, the operator λB (weighted backward shift) is hypercyclic for any $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. A related result which came later due to Kitai [24] and Gethner and Shapiro [16] is that, in addition, the set of hypercyclic vectors is G_δ and dense in $\ell^2(\mathbb{N})$. Further results on hypercyclic operators are described in [20].

In this paper we represent the new classes of hypercyclic composition operators on spaces of analytic functions. In Section 1 we consider some examples of hypercyclic composition operators on $H(\mathbb{C})$. In Section 2 we find hypercyclic composition operators on $H(\mathbb{C}^n)$ which can not be described by formula (1.1) but can be obtained from the translation operator using polynomial automorphisms of $\mathbb{C}^n$. To do it we developed a method which involves the theory of symmetric analytic functions on Banach spaces. In the first subsection we discuss some relationship between polynomial automorphisms on $\mathbb{C}^n$ and the operation of changing of polynomial bases in an algebra of symmetric analytic functions on the Banach space of summing sequences, ℓ_1. We also consider operators of the form $C_\Theta{}^{-1} T_b C_\Theta$ for a polynomial automorphism Θ and show that if C_S is a hypercyclic operator for some affine automorphism S on $\mathbb{C}^n$, then there exists a representation of the form $S = \Theta \circ (I + b) \circ \Theta^{-1} + a$ that is we can write $C_S = C_\Theta{}^{-1} T_b C_\Theta T_a$. To do it we use the method of symmetric polynomials on ℓ_1 as an important tool for constructing and computations. In the next subsection we prove the hypercyclicity of a special operator on an algebra of symmetric analytic functions on ℓ_1 which plays the role of translation in this algebra. In Section 3 we propose a simple method how to construct analytic hypercyclic operator on Fréchet spaces and Banach spaces. There are some examples. Some hypercyclic operators on spaces of analytic functions on some algebraic manifolds are described in Section 4.

For details of the theory of analytic functions on Banach spaces we refer the reader to Dineen's book [13]. Note that an analogue of the Godefroy-Shapiro Theorem for a special class of entire functions on Banach space with separable dual was proved by Aron and Bés in [2]. Current state of theory of symmetric analytic functions on Banach spaces can be found in [1, 18]. Detailed information about hypercyclic operators is given in [3].

2. Topological Transitive, Chaotic and Mixing Composition Operators

This chapter provides an introduction to the theory of hypercyclicity. Fundamental concepts such as topologically transitive, chaotic and mixing maps are defined. The Birkhoff transitivity theorem is derived as a crucial tool for showing that a map has a dense orbit.

Definition 2.1. Let X be metric space. A continuous map $T : X \to X$ is called *topologically transitive* if, for any pair U, V of nonempty open subsets of X, there exists some $n \geq 0$ such that $T^n(U) \cap V \neq \emptyset$.

Topological transitivity can be interpreted as saying that T connects all nontrivial parts of X. This is automatically the case whenever there is a point $x \in X$ with dense orbit under T. What is less obvious is that, in separable complete metric spaces, the converse of this case is also true: topologically transitive maps must have a dense orbit. This result was first obtained in 1920 by G. D. Birkhoff in the context of maps on compact subsets of $\mathbb{R}^N$.

Theorem 2.2. (**Birkhoff transitivity theorem**). *Let T be a continuous map on a separable complete metric space X without isolated points. Then the following assertions are equivalent:*
(i) T is topologically transitive;
(ii) there exists some $x \in X$ such that $\mathrm{Orb}(x, T)$ is dense in X. If one of these conditions holds then the set of points in X with dense orbit is a dense G_δ-set.

Definition 2.3. Let T be a continuous map on a metric space X.
(a) A point $x \in X$ is called a *fixed point* of T if $Tx = x$.
(b) A point $x \in X$ is called a *periodic point* of T if there is some $n \geq 1$ such that $T^n x = x$. The least such number n is called the period of x.

Definition 2.4. (**Devaney chaos**). Let X be metric space. A continuous map $T : X \to X$ is said to be *chaotic* (in the sense Devaney) if it satisfies the following conditions:
(i) T is topologically transitive;
(ii) T has a dense set of periodic points.

Definition 2.5. Let X be metric space. A continuous map $T : X \to X$ is called *mixing* if, for any pair U, V of nonempty open subsets of X, there exists some $N \geq 0$ such that

$$T^n(U) \cap V \neq \emptyset \quad \text{for all} \quad n \geq N.$$

Every mathematical theory has its notion of isomorphism. Let X, Y be metric space. When do we want to consider two continuous operators $S : Y \to Y$ and $T : X \to X$ as equal? There should be a homeomorphism $\phi : Y \to X$ such that, when $x \in X$ corresponds to $y \in Y$ via ϕ then Tx should correspond to Sy via ϕ. In other words, if $x = \phi(y)$ then $Tx = \phi(Sy)$. This is equivalent to saying that $T \circ \phi = \phi \circ S$.

We recall that a homeomorphism is a bijective continuous map whose inverse is also continuous. It is already enough to demand that ϕ is continuous with dense range.

Definition 2.6. Let X, Y be metric space and $S : Y \to Y$, $T : X \to X$ be a continuous map.
(a) Then T is called *quasiconjugate* to S if there exists a continuous map $\phi : Y \to X$ with dense range such that $T \circ \phi = \phi \circ S$, that is, the diagramm

$$\begin{array}{ccc} Y & \xrightarrow{S} & Y \\ \phi \downarrow & & \downarrow \phi \\ X & \xrightarrow{T} & X \end{array}$$

commutes.
(b) If ϕ can be chosen to be a homeomorphism then S and T are called *conjugate*.

As we seen, operators may often be interpreted in various ways. MacLane's operator is both a differential operator and a weighted shift. Birkhoff's operators are differential operators as well. Here now we have interpretation of Birkhoff's operators T_a: they are special composition operators. Writing

$$\tau_a(z) = z + a$$

we see that τ_a is an entire function such that

$$T_a f = f \circ \tau_a.$$

In fact, τ_a is even an automorphism of $\mathbb{C}$, that is, a bijective entire function. This observations serve as the starting point of another major investigation: the hypercyclicity of general composition operators.

The further results of this section we also can find in [21].

Let Ω be an arbitrary domain in $\mathbb{C}$, that is, a nonempty connected open set. An automorphism of Ω is a bijective analytic function

$$\varphi : \Omega \to \Omega$$

its inverse is then also holomorphic. The set of all automorphisms of Ω is denoted by $Aut(\Omega)$. Now, for $\varphi \in Aut(\Omega)$ the corresponding composition operator is defined as

$$C_\varphi f = f \circ \varphi,$$

that is, $(C_\varphi f)(z) = f(\varphi(z))$, $z \in \Omega$.

Definition 2.7. Let Ω be a domain in $\mathbb{C}$ and $\varphi_n : \Omega \to \Omega$, $n \geq 1$, holomorphic maps. Then the sequence $(\varphi_n)_n$ is called a *run-away sequence* if, for any compact subset $K \subset \Omega$, there is some $n \in \mathbb{N}$ such that $\varphi_n(K) \cap K = \emptyset$.

We will usually apply this definition to the sequence $(\varphi^n)_n$ of iterates of an automorphism φ on Ω. Let us consider two examples.

Example 2.8. (a) Let $\Omega = \mathbb{C}$. Then the automorphisms of $\mathbb{C}$ are the functions

$$\varphi(z) = az + b, \quad a \neq 0, \quad b \in \mathbb{C},$$

and $(\varphi^n)_n$ is run-away if and only if $a = 1$, $b \neq 0$.

Indeed, let φ be an automorphism of $\mathbb{C}$. If φ is not a polynomial then, by the Casorati-Weierstrass theorem, $\varphi(\{z \in \mathbb{C}; |z| > 1\})$ is dense in $\mathbb{C}$ and therefore intersects the set $\varphi(\mathbb{D})$, which is open by the open mapping theorem. Since this contradicts injectivity, φ must be a polynomial. Again by injectivity, its degree must be one, so that φ is of the stated form. Now, if $a = 1$ then $\varphi^n(z) = z + nb$, so that we have the run-away property if and only if $b \neq 0$; while if $a \neq 1$ then $(1-a)^{-1}b$ is a fixed point of φ so that $(\varphi^n)_n$ cannot be run-away.

(b) Let $\Omega = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, the punctured plane. An argument as in (a) shows that the automorphisms of $\mathbb{C}^*$ are the functions

$$\varphi(z) = az \quad \text{or} \quad \varphi(z) = \frac{a}{z}, \quad a \neq 0.$$

Then $(\varphi^n)_n$ is run-away if and only if $\varphi(z) = az$ with $|a| \neq 1$.

We first show that the run-away property is a necessary condition for the hypercyclicity of the composition operator.

Proposition 2.9. *Let Ω be a domain in $\mathbb{C}$ and $\varphi \in Aut(\Omega)$. If C_φ is hypercyclic then $(\varphi^n)_n$ is a run-away sequence.*

Corollary 2.10. *There is no automorphism of $\mathbb{C}^*$ whose composition operator is hypercyclic.*

If $\Omega = \mathbb{C}$, the automorphisms are given by

$$\varphi(z) = az + b, \quad a \neq 0, \quad b \in \mathbb{C},$$

and C_φ is hypercyclic if and only if $a = 1$, $b \neq 0$; see Example 2.8(a). Thus the hypercyclic composition operators on $\mathbb{C}$ are precisely Birkhoff's translation operators.

Let us now consider the simply connected domains Ω other than $\mathbb{C}$. By the Riemann mapping theorem, Ω is conformally equivalent to the unit disk, that is, there is a conformal map $\varphi : \mathbb{D} \to \Omega$. It suffices to study the case when $\Omega = \mathbb{D}$.

Proposition 2.11. *The automorphisms of $\mathbb{D}$ are the linear fractional transformations*

$$\varphi(z) = b\frac{a-z}{1-\bar{a}z}, \quad |a| < 1, |b| = 1.$$

Moreover, φ maps $\mathbb{T}$ bijectively onto itself, where $\mathbb{T}$ is the unit circle.

Now, linear fractional transformations are a very well understood class of analytic maps. Using their properties it is not difficult to determine the dynamical behaviour of the corresponding composition operators; via conjugacy these results can then be carried over to arbitrary simply connected domains.

Theorem 2.12. *Let Ω be a simply connected domain and $\varphi \in Aut(\Omega)$. Then the following assertions are equivalent:*
(i) C_φ is hypercyclic;
(ii) C_φ is mixing;
(iii) C_φ is chaotic;
(iv) $(\varphi^n)_n$ is a run-away sequence;
(v) φ has no fixed point in Ω;
(vi) C_φ is quasiconjugate to a Birkhoff's operator.

2.1. Composition Operators on the Hardy Space

In this section we consider an interesting generalization of the backward shift operator. The underlying space will be the Hardy space H^2. Arguably its easiest definition is the following. If $(a_n)_{n\geq 0}$ is a complex sequence such that

$$\sum_{n=0}^{\infty} |a_n|^2 < \infty,$$

then it is, in particular, bounded, and hence

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \mathbb{C}, |z| < 1,$$

defines a analytic function on the complex unit disk $\mathbb{D}$. The Hardy space is then defined as the space of these functions, that is,

$$H^2 = \left\{ f : \mathbb{D} \to \mathbb{C}; f(z) = \sum_{n=0}^{\infty} a_n z^n, z \in \mathbb{D}, \text{with} \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}.$$

In other words, the Hardy space is simply the sequence space $\ell^2(\mathbb{N}_0)$, with its elements written as analytic functions. It is then clear that H^2 is a Banach space under the norm

$$\|f\| = \left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{\frac{1}{2}} \quad \text{when} f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

and it is even a Hilbert space under the inner product

$$\langle f, g \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n} \quad \text{when} f(z) = \sum_{n=0}^{\infty} a_n z^n, g(z) = \sum_{n=0}^{\infty} b_n z^n.$$

The polynomials form a dense subspace of H^2.

Let φ be an automorphism of the unit disk $\mathbb{D}$ and let $C_\varphi f = f \circ \varphi$ be the corresponding composition operator, where we now demand that f belongs to H^2.

Proposition 2.13. *For any $\varphi \in Aut(\mathbb{D})$, C_φ defines an operator on H^2.*

Our aim now is to characterize when C_φ is hypercyclic on H^2. It will be convenient to consider φ as a particular linear fractional transformation.

Indeed, let

$$\varphi(z) = \frac{az+b}{cz+d}, \quad ad - bc \neq 0,$$

be an arbitrary linear fractional transformation, which we consider as a map on the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Then φ has either one or two fixed points in $\widehat{\mathbb{C}}$, or it is the identity.

Theorem 2.14. *Let $\varphi \in Aut(\mathbb{D})$ and C_φ be the corresponding composition operator on H^2. Then the following assertions are equivalent:*
(i) C_φ is hypercyclic;
(ii) C_φ is mixing;
(iii) φ has no fixed point in $\mathbb{D}$.

3. Hypercyclic Composition Operator on Space of Symmetric Functions

In this section we consider hypercyclic composition operators on space of symmetric analytic functions, the basic results are given in [31].

3.1. Polynomial Automorphisms and Symmetric Functions

Definition 3.1. A polynomial map $\Phi = (\Phi_1, \ldots, \Phi_n)$ from $\mathbb{C}^n$ to $\mathbb{C}^n$ is said to be a *polynomial automorphism* if it is invertible and the inverse map is also a polynomial.

Definition 3.2. Let X be a Banach space with a symmetric basis $(e_i)_{i=1}^\infty$. A function g on X is called *symmetric* if for every $x = \sum_{i=1}^{\infty} x_i e_i \in X$,

$$g(x) = g\Big(\sum_{i=1}^{\infty} x_i e_i\Big) = g\Big(\sum_{i=1}^{\infty} x_i e_{\sigma(i)}\Big)$$

for an arbitrary permutation σ on the set $\{1, ..., m\}$ for any positive integer m.

Definition 3.3. The sequence of homogeneous polynomials $(P_j)_{j=1}^\infty$, $\deg P_k = k$ is called a *homogeneous algebraic basis* in the algebra of symmetric polynomials if for every symmetric polynomial P of degree n on X there exists a polynomial q on $\mathbb{C}^n$ such that

$$P(x) = q(P_1(x), \ldots, P_n(x)).$$

Throughout this paper we consider the case when $X = \ell_1$. We denote by $\mathcal{P}_s(\ell_1)$ the algebra of all symmetric polynomials on ℓ_1. The next two algebraic bases of $\mathcal{P}_s(\ell_1)$ are useful for us: $(F_k)_{k=1}^\infty$ (see [18]) and $(G_k)_{k=1}^\infty$, where

$$F_k(x) = \sum_{i=1}^{\infty} x_i^k \quad \text{and} \quad G_k(x) = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}.$$

By the Newton formula $G_1 = F_1$ and for every $k > 1$,

$$G_{k+1} = \frac{1}{k+1}((-1)^k F_{k+1} - F_k G_1 + \cdots + F_1 G_k).$$

Denote by $H_s^n(\ell_1)$ the algebra of entire symmetric functions on ℓ_1 which is topologically generated by polynomials $F_1, \ldots, F_n$. It means that $H_s^n(\ell_1)$ is the completion of the algebraic span of $F_1, \ldots, F_n$ in the uniform topology on bounded subsets. We say that polynomials $P_1, \ldots, P_n$ (not necessary homogeneous) form an *algebraic basis* in $H_s^n(\ell_1)$ if they topologically generate $H_s^n(\ell_1)$. Evidently, if $(P_j)_{j=1}^{\infty}$ is a homogeneous algebraic basis in $\mathcal{P}_s(\ell_1)$, then $(P_1, \ldots, P_n)$ is an algebraic basis in $H_s^n(\ell_1)$. We will use notations $\mathbf{F} := (F_k)_{k=1}^n$ and $\mathbf{G} := (G_k)_{k=1}^n$.

Proposition 3.4. *Let $\Phi = (\Phi_1, \ldots, \Phi_n)$ be a polynomial automorphism on $\mathbb{C}^n$. Then $(\Phi_1(\mathbf{P}), \ldots, \Phi_n(\mathbf{P}))$ is an algebraic basis in $H_s^n(\ell_1)$ for an arbitrary algebraic basis $\mathbf{P} = (P_1, \ldots, P_n)$.*

Conversely, if $(\Phi_1(\mathbf{P}), \ldots, \Phi_n(\mathbf{P}))$ is an algebraic basis for some algebraic basis $\mathbf{P} = (P_1, \ldots, P_n)$ in $H_s^n(\ell_1)$ and a polynomial map Φ on $\mathbb{C}^n$, then Φ is a polynomial automorphism.

Proof. Suppose that Φ is a polynomial automorphism and

$$\Phi^{-1} = ((\Phi^{-1})_1, \ldots, (\Phi^{-1})_n)$$

is its inverse. Then $P_k = (\Phi^{-1})_k(\Phi_1(\mathbf{P}), \ldots, \Phi_n(\mathbf{P})), 1 \le k \le n$. Hence polynomials $\Phi_1(\mathbf{P}), \ldots, \Phi_n(\mathbf{P})$ topologically generate $H_s^n(\ell_1)$ and so they form an algebraic basis.

Let now $(\Phi_1(\mathbf{P}), \ldots, \Phi_n(\mathbf{P}))$ be an algebraic basis in $H_s^n(\ell_1)$ for some algebraic basis $\mathbf{P} = (P_1, \ldots, P_n)$. Then for each P_k, $1 \le k \le n$, there exists a polynomial q_k on $\mathbb{C}^n$ such that $P_k = q_k(\Phi_1(\mathbf{P}), \ldots, \Phi_n(\mathbf{P}))$. Put $(\Phi^{-1})_k(t) := q_k(t)$, $t \in \mathbb{C}^n$. Since $(\Phi_1(\mathbf{P}), \ldots, \Phi_n(\mathbf{P}))$ is an algebraic basis, the map

$$(x_1, \ldots, x_n) \mapsto (\Phi_1(\mathbf{P}(x)), \ldots, \Phi_n(\mathbf{P}(x)))$$

is onto by [1, Lemma 1.1]. Thus $\Phi\colon \mathbb{C}^n \to \mathbb{C}^n$ is a bijection and so the mapping $((\Phi^{-1})_1, \ldots, (\Phi^{-1})_n)$ is the inverse polynomial map for Φ. □

3.2. SIMILAR TRANSLATIONS

We start with an evident statement, which actually is a very special case of the Universal Comparison Principle (see e.g. [20, Proposition 4]).

Proposition 3.5. *Let T be a hypercyclic operator on X and A be an isomorphism of X. Then $A^{-1}TA$ is hypercyclic.*

We will say that $A^{-1}TA$ is a *similar* operator to T. If $T = C_R$ is a composition operator on $H(\mathbb{C}^n)$ and $A = C_\Phi$ is a composition by an analytic automorphism Φ of $\mathbb{C}^n$, then $A^{-1}TA = C_{\Phi\circ R\circ\Phi^{-1}}$ is a composition operator too. If A is a composition with a polynomial automorphism, we will say that $A^{-1}TA$ is *polynomially similar* to T. Now we consider operators which are similar to the translation composition $T_a\colon f(x) \mapsto f(x+a)$ on $H(\mathbb{C}^n)$.

Example 3.6. Let $\Phi(t_1, t_2) = (t_1, t_2 - t_1^m)$ for some positive integer m. Clearly, Φ is a polynomial automorphism and $\Phi^{-1}(z_1, z_2) = (z_1, z_2 + z_1^m)$. So

$$\begin{aligned}\Phi(t+a) &= (t_1 + a_1, t_2 + a_2 - (t_1 + a_1)^m) \\ &= \left(t_1 + a_1, t_2 + a_2 - t_1^m - a_1^m - \sum_{j=1}^{m-1} \binom{m-j}{j} t_1^{m-j} a_1^j\right).\end{aligned}$$

Thus we have

$$\Phi \circ (I+a) \circ \Phi^{-1}(t) = \Phi(\Phi^{-1}(t)+a) = \left(t_1 + a_1, t_2 + a_2 - a_1^m - \sum_{j=1}^{m-1} \binom{m-j}{j} t_1^{m-j} a_1^j\right).$$

Hence the composition operator with the $(m-1)$-degree polynomial $\Phi \circ (I+a) \circ \Phi^{-1}$ is similar to the translation operator $T_a = C_{(I+a)}$ and so it must be hypercyclic. Here I is the identity operator.

It is known (see [1]) that the map

$$\mathcal{F}_n^{\mathbf{F}} \colon f(t_1, \ldots, t_n) \mapsto f(F_1(x), \ldots, F_n(x))$$

is a topological isomorphism from the algebra $H(\mathbb{C}^n)$ to the algebra $H_s^n(\ell_1)$. Now we will prove more general statement.

Lemma 3.7. *Let $\mathbf{P} = (P_k)_{k=1}^n$ be an algebraic basis in $H_s^n(\ell_1)$. Then the map*

$$\mathcal{F}_n^{\mathbf{P}} \colon f(t_1, \ldots, t_n) \mapsto f(P_1(x), \ldots, P_n(x))$$

is a topological isomorphism from $H(\mathbb{C}^n)$ onto $H_s^n(\ell_1)$.

Proof. Evidently, $\mathcal{F}_n^{\mathbf{P}}$ is a homomorphism. It is known [1] that for every vector $(t_1, \ldots, t_n) \in \mathbb{C}^n$ there exists an element $x \in \ell_1$ such that $P_1(x) = t_1, \ldots, P_n(x) = t_n$. Therefore the map $\mathcal{F}_n^{\mathbf{P}}$ is injective. Let us show that $\mathcal{F}_n^{\mathbf{P}}$ is surjective. Let $u \in H_s^n(\ell_1)$ and $u = \sum u_k$ be the Taylor series expansion of u at zero. For every homogeneous polynomial u_k there exists a polynomial q_k on $\mathbb{C}^n$ such that $u_k = q_k(P_1, \ldots, P_n)$. Put $f(t_1, \ldots, t_n) = \sum_{k=1}^{\infty} q_k(t_1, \ldots, t_n)$. Since f is a power series which converges for every vector $(t_1, \ldots, t_n)$, f is an entire analytic function on $\mathbb{C}^n$. Evidently, $\mathcal{F}_n^{\mathbf{P}}(f) = u$. From the known theorem about automatic continuity of an isomorphism between commutative finitely generated Fréchet algebras [23, p. 43] it follows that $\mathcal{F}_n^{\mathbf{P}}$ is continuous. □

Let $x, y \in \ell_1$, $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$. We put

$$x \bullet y := (x_1, y_1, x_2, y_2, \ldots)$$

and define

$$\mathcal{T}_y(f)(x) := f(x \bullet y).$$

We will say that $x \mapsto x \bullet y$ is the *symmetric translation* and the operator $\mathcal{T}_y$ is the *symmetric translation operator*. It is clear that if f is a symmetric function, then $f(x \bullet y)$ is a symmetric function for any fixed y.

In [12] is proved that $\mathcal{T}_y$ is a topological isomorphism from the algebra of symmetric analytic functions to itself. Evidently, we have that

$$F_k(x \bullet y) = F_k(x) + F_k(y) \tag{3.1}$$

for every k.

Let $g \in H_s^n(\ell_1)$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$. Set

$$\mathcal{D}^\alpha g := \mathcal{F}_n^{\mathbf{F}} D^\alpha (\mathcal{F}_n^{\mathbf{F}})^{-1} g = \left(\frac{\partial^{\alpha_1}}{\partial t_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial t_n^{\alpha_n}} f\right)(F_1(\cdot), \ldots, F_n(\cdot)),$$

where $f = (\mathcal{F}_n^{\mathbf{F}})^{-1} g$.

Theorem 3.8. *Let $y \in \ell_1$ such that $(F_1(y), \dots, F_n(y))$ is a nonzero vector in $\mathbb{C}^n$. Then the symmetric translation operator $\mathcal{T}_y$ is hypercyclic on $H_s^n(\ell_1)$. Moreover, every operator $\mathcal{A}$ on $H_s^n(\ell_1)$ which commutes with $\mathcal{T}_y$ and is not a scalar multiple of the identity is hypercyclic and can be represented by*

$$\mathcal{A}(g) = \sum_{|\alpha| \geq 0} c_\alpha \mathcal{D}^\alpha g, \tag{3.2}$$

where c_α are coefficients of a non-constant entire function of exponential type on $\mathbb{C}^n$.

Proof. Let $a = (F_1(y), \dots, F_n(y)) \in \mathbb{C}^n$. If $g \in H_s^n(\ell_1)$, then

$$g(x) = \mathcal{F}_n^{\mathbf{F}}(f)(x) = f(F_1(x), \dots, F_n(x))$$

for some $f \in H_s^n(\ell_1)$ and property (3.1) implies that

$$\mathcal{T}_y(g)(x) = \mathcal{F}_n^{\mathbf{F}} T_a (\mathcal{F}_n^{\mathbf{F}})^{-1}(g)(x).$$

So the proof follows from Proposition 3.5 and the Godefroy-Shapiro Theorem. □

A given algebraic basis $\mathbf{P}$ on $H_s^n(\ell_1)$ we set

$$T_{\mathbf{P},y} := (\mathcal{F}_n^{\mathbf{P}})^{-1} \mathcal{T}_y \mathcal{F}_n^{\mathbf{P}} \quad \textit{and} \quad D_{\mathbf{P}}^{\alpha} := (\mathcal{F}_n^{\mathbf{P}})^{-1} \mathcal{D}^\alpha \mathcal{F}_n^{\mathbf{P}}.$$

Corollary 3.9. *Let $\mathbf{P}$ be an algebraic basis on $H_s^n(\ell_1)$ and let $y \in \ell_1$ such that $(F_1(y), \dots, F_n(y)) \neq 0$. Then the operator $T_{\mathbf{P},y}$ is hypercyclic on $H(\mathbb{C}^n)$. Moreover, every operator A on $H(\mathbb{C}^n)$ which commutes with $T_{\mathbf{P},y}$ and is not a scalar multiple of the identity is hypercyclic and can be represented by the form*

$$A(f) = \sum_{|\alpha| \geq 0} c_\alpha D_{\mathbf{P}}^{\alpha} f, \tag{3.3}$$

where c_α as in (1.1).

Note that due to Proposition 3.4 the transformation $(\mathcal{F}_n^{\mathbf{P}})^{-1} \mathcal{T}_y \mathcal{F}_n^{\mathbf{P}}$ is nothing else than a composition with $\Phi \circ (I + a) \circ \Phi^{-1}$, where $\Phi(F_1, \dots, F_n) = (P_1, \dots, P_n)$ and $a = (F_1(y), \dots, F_n(y))$. Conversely, every polynomially similar operator to the translation can be represented by the form $(\mathcal{F}_n^{\mathbf{P}})^{-1} \mathcal{T}_y \mathcal{F}_n^{\mathbf{P}}$ for some algebraic basis of symmetric polynomials $\mathbf{P}$. This observation can be helpful in order to construct some examples of such operators.

Example 3.10. Let us compute how looks the operator $T_{\mathbf{P},y}$ for $\mathbf{P} = \mathbf{G}$. We observe first that $G_k(x \bullet y) = \sum_{i=0}^{k} G_i(x) G_{k-i}(y)$, where for the sake of convenience we take $G_0 \equiv 1$. Thus

$$\begin{aligned} \mathcal{T}_y \mathcal{F}_n^{\mathbf{G}} f(t_1, \dots, t_n) &= \mathcal{T}_y f(G_1(x), \dots, G_n(x)) = f(G_1(x \bullet y), \dots, G_n(x \bullet y)) \\ &= f\Big(G_1(x) + G_1(y), \dots, \sum_{i=0}^{n} G_i(x) G_{n-i}(y)\Big). \end{aligned}$$

Therefore

$$T_{\mathbf{G},y} f(t_1, \dots, t_n) = f\Big(t_1 + b_1, \dots, \sum_{i=0}^{k} t_i b_{k-i}, \dots, \sum_{i=0}^{n} t_i b_{n-i}\Big), \tag{3.4}$$

where $t_0 = 1$, $b_0 = 1$ and $b_k = G_k(y)$ for $1 \leq k \leq n$.

According to the Newton formula and Proposition 3.4 the corresponding polynomial automorphism Φ can be given of recurrence form $\Phi_1(t) = t_1$, $\Phi_{k+1}(t) = 1/(k+1)((-1)^k t_{k+1} - t_k \Phi_1(t) + \cdots + t_1 \Phi_k(t))$ which is not so good for computations.

The hypercyclic operator in Example 3.6 is a composition with an $m-1$ degree polynomial and so does not commute with the translation because it can not be generated by formula (1.1). However, the composition with an affine map in Example 3.10 still does not commute with T_a. Indeed, by (3.4),

$$T_a \circ T_{\mathbf{G},y} f(t_1,\ldots,t_n) = f\Big(t_1+b_1+a_1,\ldots,\sum_{i=0}^{n} t_i b_{n-i} + a_n\Big);$$

$$T_{\mathbf{G},y} \circ T_a f(t_1,\ldots,t_n) = f\Big(t_1+b_1+a_1,\ldots,\sum_{i=0}^{n} (t_i+a_i) b_{n-i}\Big),$$

where $a_0 = 1$. Evidently, $T_a \circ T_{\mathbf{G},y} \neq T_{\mathbf{G},y} \circ T_a$ for some $a \neq 0$ whenever $b \neq (0,\ldots,0,b_n)$.

Corollary 3.11. *There exists a nonzero vector $b \in \mathbb{C}^n$ and a polynomial automorphism Θ on $\mathbb{C}^n$ such that $\Theta \circ (I+b)\Theta^{-1}(t) = A(t) + c$ where A is a linear operator with the matrix of the form*

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 1 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \tag{3.5}$$

and $c \neq 0$.

Proof. We choose $b \in \mathbb{C}^n$ such that all coordinates b_k, $1 \le k \le n$ are positive numbers. Let Φ be a polynomial automorphism associated with $T_{\mathbf{G},y}$ in Example 3.10, where $y \in \ell_1$ is such that $G_k(y) = b_k$, $1 \le k \le n$. Then, according to (3.4), we can write $\Phi \circ (I+b)\Phi^{-1}(t) = R(t) + b$, where

$$R = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ b_1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{n-2} & b_{n-3} & \cdots & 1 & 0 \\ b_{n-1} & b_{n-2} & \cdots & b_1 & 1 \end{pmatrix}.$$

We recall that the *index* of an eigenvalue λ of a matrix M is the smallest nonnegative integer k such that $\operatorname{rank}((M-\lambda I)^k) = \operatorname{rank}((M-\lambda I)^{k+1})$. The matrix R has a unique eigenvalue 1 and since all coordinates b_k of b are positive, the index of this eigenvalue is equal to n. Indeed, for each $k < n$, $(R-\lambda I)^k$ contains an $(n-k)\times(n-k)$ triangular matrix with only positive numbers in the main diagonal and $(R-\lambda I)^n = 0$. Therefore, from the Linear Algebra we know that the largest Jordan block A associated with the eigenvalue 1 is $n \times n$ and so it can be represented by (3.5). Thus there is a linear isomorphism L on $\mathbb{C}^n$ such that $A = LRL^{-1}$. Hence

$$(L\circ\Phi)\circ(I+b)\circ(L\circ\Phi)^{-1}(t) = L\circ(R+b)\circ L^{-1}(t) = A(t) + L(b).$$

So it is enough to set $\Theta := L \circ \Phi$ and $c := L(b)$. □

Theorem 3.12. *Let S be an affine automorphism on $\mathbb{C}^n$ such that C_S is hypercyclic. Then there are vectors a, b and a polynomial automorphism Θ on $\mathbb{C}^n$ such that $S = \Theta \circ (I+b) \circ \Theta^{-1} + a$.*

Proof. Let $S(t) = A(t) + c$ be an affine map on $\mathbb{C}^n$ such that C_S is hypercyclic. Without loss of the generality we can assume that A is a direct sum of Jordan blocks $A_1,\ldots,A_m$ and each block A_j acts on a subspace V_j of $\mathbb{C}^n$. In the proof of Theorem 3.1 of [5] is shown that the spectrum of each block A_j is the singleton $\{1\}$. So each A_j is of the form as in (3.5). Let $\Theta_{(j)}$ be a polynomial automorphism of V_j as in Corollary 3.11, that is,

$$\Theta_{(j)} \circ (I + b_{(j)}) \circ \Theta_{(j)}^{-1} = A_j + b_{(j)},$$

for some $b_{(j)} \in V_j$. Put $\Theta = \Theta_{(1)} + \cdots + \Theta_{(m)}$ and $b = b_{(1)} + \cdots + b_{(m)}$. Then $\Theta \circ (I + b) \circ \Theta^{-1} = A + b$. Let $a = c - b$. Hence

$$S = A + c = A + b + a = \Theta \circ (I + b) \circ \Theta^{-1} + a.$$

□

Of course, the converse of Theorem 3.12 (with $b \neq 0$) also holds.

We do not know whether it is always possible to choose Θ so that $a = 0$. In other words: *Is every hypercyclic operator which is a composition by an affine automorphism polynomially similar to a translation?* Moreover, we do not know any example of a hypercyclic composition operator on $H(\mathbb{C}^n)$ which is not similar to a translation.

3.3. The Infinity-Dimensional Case

Let us recall a well known Kitai-Gethner-Shapiro theorem which is also known as the Hypercyclicity Criterion.

Theorem 3.13. *Let X be a separable Fréchet space and $T\colon X \to X$ be a linear and continuous operator. Suppose there exist X_0, Y_0 dense subsets of X, a sequence (n_k) of positive integers and a sequence of mappings (possibly nonlinear, possibly not continuous) $S_n\colon Y_0 \to X$ so that*

(1) $T^{n_k}(x) \to 0$ *for every* $x \in X_0$ *as* $k \to \infty$.
(2) $S_{n_k}(y) \to 0$ *for every* $y \in Y_0$ *as* $k \to \infty$.
(3) $T^{n_k} \circ S_{n_k}(y) = y$ *for every* $y \in Y_0$.

Then T is hypercyclic.

The operator T is said to satisfy the *Hypercyclicity Criterion for full sequence* if we can chose $n_k = k$. Note that T_a satisfies the Hypercyclicity Criterion for full sequence [17] and so the symmetric shift $\mathcal{T}_y$ on $H_s^n(\ell_1)$ satisfies the Hypercyclicity Criterion for full sequence provided $(F_1(y), \ldots, F_n(y)) \neq 0$.

Finally, we establish our result about hypercyclic operators on the space of symmetric entire functions on ℓ_1. But before this, we need the following general auxiliary statement, which might be of some interest by itself.

Lemma 3.14. *Let X be a Fréchet space and $X_1 \subset X_2 \subset \cdots \subset X_m \subset \cdots$ be a sequence of closed subspaces such that $\bigcup_{m=1}^{\infty} X_m$ is dense in X. Let T be an operator on X such that $T(X_m) \subset X_m$ for each m each restriction $T|_{X_m}$ satisfies the Hypercyclicity Criterion for full sequence on X_m. Then T satisfies the Hypercyclicity Criterion for full sequence on X.*

Proof. Let $Y_0^{(m)}$ and $X_0^{(m)}$ be dense subsets in X_m, and $S_k^{(m)}$ corresponding sequence of mappings associated with $T|_{X_m}$ as in Theorem 3.13. Put $X_0 = \bigcup_{m=1}^{\infty} X_0^{(m)}$ and $Y_0 = \bigcup_{m=1}^{\infty} Y_0^{(m)}$. It is clear that both X_0 and Y_0 are dense in X. For a given $y \in Y_0$, we denote by $m(y)$ the minimal number m such that $y \in Y_0^{(m)}$. We set $S_k(y) := S_k^{(m(y))}(y)$. Then

$$T^k \circ S_k(y) = T^k|_{X_{m(y)}} \circ S_k^{(m(y))}(y) = y, \quad \forall y \in Y_0$$

and $S_k(y) = S_k^{(m(y))}(y) \to 0$ as $k \to \infty$ for every $y \in Y_0$. Similarly, if $x \in X_0$, then $x \in X_0^{(m)}$ for some m and $T^k(x) = T^k|_{X_m}(x) \to 0$ as $k \to \infty$. So T satisfies the Hypercyclicity Criterion for full sequence on X. In particular, T is hypercyclic. □

We denote by $H_{bs}(\ell_1)$ the Fréchet algebra of symmetric entire functions on ℓ_1 which are bounded on bounded subsets. This algebra is the completion of the space of symmetric polynomials on ℓ_1 endowed with the uniform topology on bounded subsets.

Theorem 3.15. *The symmetric operator $\mathcal{T}_y$ is hypercyclic on $H_{bs}(\ell_1)$ for every $y \neq 0$.*

Proof. Since $y \neq 0$, $F_{m_0}(y) \neq 0$ for some m_0 [1]. So, $\mathcal{T}_y$ is hypercyclic (and satisfies the Hypercyclicity Criterion for full sequence) on $H_s^m(\ell_1)$ whenever $m \geq m_0$. The set $\bigcup_{m=m_0}^{\infty} H_s^m(\ell_1)$ contains the space of all symmetric polynomials on ℓ_1 and so it is dense in $H_{bs}(\ell_1)$. Also $H_s^m(\ell_1) \subset H_s^n(\ell_1)$ if $n > m$. Hence by Lemma 3.14, $\mathcal{T}_y$ is hypercyclic. □

4. Analytic Hypercyclic Operators

In this section we will show a simple method how to construct polynomial and analytic hypercyclic operators. Basic results of this section we can find in [29].

Let F be an analytic automorphism of X onto X and T be an hypercyclic operator on X. Then $T_F := FTF^{-1}$ (and $T_{F^{-1}} := F^{-1}TF$ as well) must be hypercyclic [20] and, in the general case, they are nonlinear. The following examples show that T_F are nonlinear for some well known hypercyclic operators T and simple analytic automorphisms F.

Example 4.1. Let $A(D)$ be the disk-algebra of all analytic functions on the unit disk D of $\mathbb{C}$ which are continuous on the closure $\overline{D}$. Denote $X_1 = \{\sum\limits_{k=0}^{\infty} a_{2k+1}t^{2k+1} \in A(D)\}$ and $X_2 = \{\sum\limits_{k=0}^{\infty} a_{2k}t^{2k} \in A(D)\}$. Clearly $A(D) = X_1 \oplus X_2$.

For every $f = f_1 + f_2$, $f_1 \in X_1$, $f_2 \in X_2$ we put

$$\begin{cases} F(f_1) & := & f_1, \\ F(f_2) & := & f_2 + f_1^2. \end{cases} \qquad \text{Then we have} \qquad \begin{cases} F^{-1}(f_1) & = & f_1, \\ F^{-1}(f_2) & = & f_2 - f_1^2. \end{cases}$$

So F is a polynomial automorphism of X. Let $T(f(t)) = f(\frac{t+1}{2})$. It is known that T is hypercyclic on $A(D)$ [10, p. 4].

Let us show that $T_F = FTF^{-1}$ is nonlinear. It is enough to check that $T_F(\lambda f) \neq \lambda T_F(f)$ for some $\lambda \in \mathbb{C}$ and $f \in A(D)$. Let $f(t) = t + t^2 \in A(D)$. Then

$$\begin{aligned} T_F(\lambda f) &= F(T(F^{-1}(\lambda t + \lambda t^2))) = F(T(\lambda t + \lambda t^2 - \lambda^2 t^2)) \\ &= F(T(\lambda t + (\lambda - \lambda^2)t^2)) = F\Big(\lambda\Big(\frac{t+1}{2}\Big) + (\lambda - \lambda^2)\Big(\frac{t+1}{2}\Big)^2\Big) \\ &= \frac{(2\lambda - \lambda^2)t}{2} + \frac{(\lambda + 3\lambda^2 - 4\lambda^3 + \lambda^4)t^2}{4} + \frac{(3\lambda - \lambda^2)}{4} \end{aligned}$$

for any $\lambda \neq 0$, $\lambda \neq 1$. Thus $T_F(\lambda f) \neq \lambda T_F(f)$.

By the similar way in the next example we consider the space of entire analytic functions $H(\mathbb{C})$ and $T(f) = f(x + a)$ to show that $T_{F^{-1}}$ is nonlinear, where F is defined as above.

Example 4.2. Let $f(t) = t + t^2 \in H(\mathbb{C})$ then $F(f) = t + 2t^2$, $F(\lambda f) = \lambda(t+t^2) + \lambda^2 t^2$. Thus

$$T(F(\lambda f)) = \lambda(t+a) + 2\lambda(1+\lambda)at + (\lambda+\lambda^2)(t^2+a^2)$$

for any $\lambda \neq 0$. Since $F^{-1}(f) = t - t^2$, we have

$$\begin{aligned} F^{-1}TF(\lambda f) &= \lambda(t+t^2) - 4\lambda^2 a^2(t+t^2) + \lambda(a+a^2+2t) + 4\lambda^2 at(t+a) \\ &\quad -4\lambda^3 at(t+a) - 4\lambda^3 a^2 t^2(2+\lambda) \neq \lambda T_{F^{-1}}(f). \end{aligned}$$

So, the operator $T_{F^{-1}} = F^{-1}TF$ is nonlinear.

Example 4.3. Next we consider the Hilbert space ℓ_2. Let $(e_k)_{k=1}^{\infty}$ be an orthonormal basis in ℓ_2 and $x = \sum_{k=1}^{\infty} x_k e_k \in \ell_2$. We define an analytic automorphism $F : \ell_2 \to \ell_2$ by the formula

$$\begin{cases} F(x_{2k-1}e_{2k-1}) &= x_{2k-1}e_{2k-1}, \\ F(x_{2k}e_{2k}) &= x_{2k}e^{-x_{2k-1}}e_{2k}, \qquad k = 1, 2, \ldots. \end{cases}$$

Let T_μ be a weighted shift

$$T_\mu(x) = (\mu x_2, \mu x_3, \ldots).$$

T_μ is a hypercyclic operator if $|\mu| > 1$ (see [34]). Then the operator $T_F = FT_\mu F^{-1}$ is hypercyclic. We will show that T_F is nonlinear.

Let $a \in \ell_2$, $a = (a_1, a_2, \ldots a_n, \ldots)$, $a = \sum_{k=1}^{\infty} a_k e_k$ and $\lambda \in \mathbb{C}$. We will show that $T_F(\lambda a) \neq \lambda T_F(a)$.

$$F^{-1}T_\mu F(\lambda a) = (\mu\lambda a_2 e^{-\lambda a_1}, \mu\lambda a_3 e^{\mu\lambda a_2 e^{-\lambda a_1}}, \mu\lambda a_4 e^{-\lambda a_3}, \mu\lambda a_5 e^{\mu\lambda a_4 e^{-\lambda a_3}}, \ldots).$$

So, $T_F(\lambda a) \neq \lambda T_F(a)$ and moreover, the map $\lambda \rightsquigarrow T_\mu(\lambda a)$ is not polynomial. Thus T_F is an analytic (not polynomial) hypercyclic map.

5. Hypercyclic Operators on Spaces of Functions on Algebraic Manifolds

In this section we represent the basic results which had obtained in [30].

Let $q_1, \ldots, q_m$ be polynomials on $\mathbb{C}^n$. We consider an ideal which is generated by the polynomials

$$\mathcal{I} = (q_1, \ldots, q_n) := \{q_1 p_1 + \cdots + q_n q_n \mid p_k \in \mathcal{P}(\mathbb{C}^n), k = 1, \ldots, n\}.$$

Let $V(\mathcal{I}) = \cap_{k=1}^{n} \ker q_k$ be set of zeros of the ideal $\mathcal{I}$. The set $V(\mathcal{I})$ is called algebraic set and on this set we can define algebra of polynomials

$$\mathcal{P}(V(\mathcal{I})) := \mathcal{P}(\mathbb{C}^n)/I(V(\mathcal{I})),$$

where $I(V(\mathcal{I}))$ is set of polynomials, which are equal to zero on $V(\mathcal{I})$.

Definition 5.1. The ideal $\mathcal{I}$ is called *simple* if from $p \in \mathcal{I}$ and $p = p_1 p_2$ follows that $p_1 \in \mathcal{I}$ and $p_2 \in \mathcal{I}$. In this case the set $V(\mathcal{I})$ is called *algebraic manifold*.

It is known from algebraic geometry (see [33]), that for simple ideal $\mathcal{I}$, $I(V(\mathcal{I})) = \mathcal{I}$, and algebra $\mathcal{P}(V(\mathcal{I})) = \mathcal{P}(\mathbb{C}^n)/I(V(\mathcal{I}))$ is ring integrity, that is ring without zero divisors. Every element of algebra $\mathcal{P}(V(\mathcal{I}))$ is class of equivalence for some $p \in \mathcal{P}(\mathbb{C}^n)$,

$$[p] = \{p + q : q \in \mathcal{I}\}.$$

We define algebra of entire analytic functions $H(V(\mathcal{I}))$ on the algebraic manifold $V(\mathcal{I})$ as set of classes

$$\{[f] : [f] = \{f + q : q \in \mathcal{I}\}, f \in H(\mathbb{C}^n)\}.$$

Let $\mathcal{N} = \{i_1, \ldots, i_k\}$ be some proper subset in $\{1, 2, \ldots, n\}$ and $\mathcal{M} = \{j_1, \ldots, j_m\} = = \{1, 2, \ldots, n\} \backslash \mathcal{N}$.
The equation

$$t_{i_1} = t_{i_2} = \cdots = t_{i_k} = 0 \tag{5.1}$$

sets in $\mathbb{C}^n$ linear subspace $L_{\mathcal{M}}$. From another side, if

$$\begin{cases} t_1 = \Phi_1(z_1, \ldots, z_n), \\ \vdots \\ t_n = \Phi_n(z_1, \ldots, z_n) \end{cases}$$

for polynomials $\Phi_1, \ldots, \Phi_n$, then equation (5.1) in coordinates $z_{i_1}, \ldots, z_{i_k}$ sets algebraic manifold $V_{\mathcal{M}}$:

$$\Phi_{i_1}(z_1, \ldots, z_n) = 0, \ \ldots, \Phi_{i_k}(z_1, \ldots, z_n) = 0.$$

By striction map $\Phi = (\Phi_1, \ldots, \Phi_n)$ on manifold $V_{\mathcal{M}}$ we get polynomial automorphism

$$\begin{cases} \Phi_{j_1}(z_1, \ldots, z_n) = t_{j_1}, \\ \vdots \\ \Phi_{j_m}(z_1, \ldots, z_n) = t_{j_m} \end{cases}$$

from $V_{\mathcal{M}}$ on $L_{\mathcal{M}}$, which we denote $\widetilde{\Phi}$.

By the another words, manifold $V_{\mathcal{M}}$ is image of subspace $L_{\mathcal{M}}$ at polynomial map $((\widetilde{\Phi}^{-1})_{j_1}, \ldots, (\widetilde{\Phi}^{-1})_{j_m})$.

Theorem 5.2. *Let a be non zero vector in $L_{\mathcal{M}}$. Then composition operator with polynomial map $\widetilde{\Phi}^{-1} \circ (I + a) \circ \widetilde{\Phi}$ is hypercyclic operator on space $H(V_{\mathcal{M}})$.*

Proof. The composition operator with translation $I + a$ is hypercyclic map. Since $\widetilde{\Phi}$ is polynomial automorphism from $V_{\mathcal{M}}$ in $L_{\mathcal{M}}$, thus $C_{\widetilde{\Phi}}$ is continuous homomorphism from $H(L_{\mathcal{M}})$ in $H(V_{\mathcal{M}})$. Then, according to Universal Comparison Principle $C_{\widetilde{\Phi}^{-1} \circ (I+a) \circ \widetilde{\Phi}} = C_{\widetilde{\Phi}} \circ T_a \circ C_{\widetilde{\Phi}^{-1}}$ is hypercyclic operator on space $H(V_{\mathcal{M}})$. $\square$

Example 5.3. Let $\Phi : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial automorphism:

$$\begin{cases} t_1 = z_1 \\ t_2 = z_2 + P(z_1), \end{cases}$$

where P is some polynomial on $\mathbb{C}$. Put $\mathcal{N} = \{2\}$, $\mathcal{M} = \{1\}$. Then

$$L_{\mathcal{M}} = \{t = (t_1, t_2) \in \mathbb{C}^2 : t_2 = 0\},$$

$$V_{\mathcal{M}} = \{z = (z_1, z_2) \in \mathbb{C}^2 : z_2 + P(z_1) = 0\}.$$

The map $\widetilde{\Phi} : V_{\mathcal{M}} \to L_{\mathcal{M}}$ is defined by formula $\widetilde{\Phi} : (z_1, z_2) \to (t_1, t_2) = (z_1, 0)$. Thus $\widetilde{\Phi}^{-1} : (t_1, 0) \to (z_1, z_2) = (t_1, -P(t_1))$. Hence, for $a = (a_1, a_2) \in L_{\mathcal{M}}$, $a_1 \neq 0$, $a_2 = 0$, automorphism

$\widetilde{\Phi}^{-1} \circ (I+a) \circ \widetilde{\Phi}$ we have a representation

$$\begin{aligned} \widetilde{\Phi}^{-1} \circ (I+a) \circ \widetilde{\Phi}(z_1, z_2) &= \widetilde{\Phi}^{-1} \circ (I+a)(z_1, 0) \\ &= \widetilde{\Phi}^{-1}(z_1 + a_1, 0) \\ &= (z_1 + a_1, -P(z_1 + a_1)), \end{aligned}$$

where I is identity operator on $\mathbb{C}^2$. Thus $C_{\widetilde{\Phi}} T_a C_{\widetilde{\Phi}^{-1}}(f)(z) = f((z_1 + a_1), -P(z_1 + a_1))$ is hypercyclic composition operator on $V_{\mathcal{M}}$.

The following questions are natural: Are there polynomials $P \in \mathcal{P}(\mathbb{C}^n)$ for which there is polynomial automorphism from $\mathbb{C}^{n-1}$ in $\ker P$? That is, for which P we can find polynomials $\Phi_1, \ldots, \Phi_n$ on $\mathbb{C}^n$ such that, the map $\Phi = (P, \Phi_2, \ldots, \Phi_n)$ is polynomial automorphism?

It is known that necessary condition for this Jacobi equality

$$\frac{\partial \Phi}{\partial t} = \begin{vmatrix} \frac{\partial P}{\partial t_1} & \cdots & \frac{\partial P}{\partial t_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \Phi_n}{\partial t_1} & \cdots & \frac{\partial \Phi_n}{\partial t_n} \end{vmatrix} \tag{5.2}$$

is equal to some non zero constant M. Denote by $Q_k^{(t)}$ minors, which are complements to 1-th array and k-th column. Evidently, $Q_k^{(t)}$ are polynomials. Expanding the determinant (5.2) along the first column, we get, that

$$\sum_{k=1}^{n} \frac{\partial P(t)}{\partial t_k} Q_k(t) = M.$$

That is, the polynomials $\frac{\partial P}{\partial t_k}$, $(k = 1, \ldots, n)$ generate ideal, which coincides with the whole space of polynomials on $\mathbb{C}^n$. Thus $\frac{\partial P}{\partial t_k}$ do not have common zeros. So we get the next proposition.

Proposition 5.4. *If there is polynomial automorphism from $\mathbb{C}^{n-1}$ in $\ker P$, then polynomials $\frac{\partial P(t)}{\partial t_k}$, $(k = 1, \ldots, n)$ do not have common zeros.*

Is it true vice versa? This question is related to the well-known Jacobi problem which remains open since 1939.

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В цій статті міститься огляд теорії гіперциклічних операторів композиції, зокрема представлено нові класи гіперциклічних операторів композиції на просторах аналітичних функцій.

Ключові слова: гіперциклічні оператори, функціональні простори, поліноміальні автоморфізми, симетричні функції.

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THE FOURTH ORDER MIXED PERIODIC RECURRENCE FRACTIONS

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Abstract. Offered economical algorithm for calculation of rational shortenings of the fourth-order mixed periodic recurrence fraction.

Keywords: parafunctions of triangular matrices, recurrence fractions, algebraic equations.

1. INTRODUCTION

The author got his idea of recurrence fractions in the year 2002 in [1], where continued fractions are written in terms of parapermanents of triangular matrices, i.e. the second-order recurrence fractions. The third-order recurrence fractions were studied by the author and his postgraduate student Semenchuk A.V. in [2],[3],[4]. This study is a natural continuation of [5]; that is why, in the present paper, we use references to the theorems and formulas of the latter. Given that, we shall denote the numbers of the formulas and theorems from [5] with a stroke.

2. The Fourth-Order Mixed Periodic Recurrence Fractions

Definition 2.1. *The 4-th order recurrence fraction*

$$\alpha=\left[\begin{array}{c|ccccccccccccccc}
q_0^* & & & & & & & & & & & & & & & \\
\frac{p_1^*}{q_1^*} & q_1^* & & & & & & & & & & & & & & \\
\frac{r_2^*}{p_2^*} & \frac{p_2^*}{q_2^*} & q_2^* & & & & & & & & & & & & & \\
\frac{s_3^*}{r_3^*} & \frac{r_3^*}{p_3^*} & \frac{p_3^*}{q_3^*} & q_3^* & & & & & & & & & & & & \\
0 & \frac{s_4^*}{r_4^*} & \frac{r_4^*}{p_4^*} & \frac{p_4^*}{q_4^*} & q_4^* & & & & & & & & & & & \\
\vdots & \dots & \dots & \dots & \dots & \ddots & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & \dots & q_{l-3}^* & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & \dots & \frac{p_{l-2}^*}{q_{l-2}^*} & q_{l-2}^* & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & \dots & \frac{r_{l-1}^*}{p_{l-1}^*} & \frac{p_{l-1}^*}{q_{l-1}^*} & q_{l-1}^* & & & & & & & \\
0 & 0 & 0 & 0 & 0 & \dots & \frac{s_l^*}{r_l^*} & \frac{r_l^*}{p_l^*} & \frac{p_l^*}{q_l^*} & q_l^* & & & & & & \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{s_1}{r_1} & \frac{r_1}{p_1} & \frac{p_1}{q_1} & q_1 & & & & & \\
\vdots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \ddots & & & & & \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & q_{k-3} & & & & \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & \frac{p_{k-2}}{q_{k-2}} & q_{k-2} & & & \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & \frac{r_{k-1}}{p_{k-1}} & \frac{p_{k-1}}{q_{k-1}} & q_{k-1} & & \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & \frac{s_k}{r_k} & \frac{r_k}{p_k} & \frac{p_k}{q_k} & q_k & \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & \frac{s_1}{r_1} & \frac{r_1}{p_1} & \frac{p_1}{q_1} & q_1 \\
\vdots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \ddots
\end{array}\right]_\infty, \tag{2.1}$$

is called the 4-th order mixed k-periodic recurrence fraction with the preperiod of l.

We shall study the fourth-order 5-periodic recurrence fractions with the preperiod of 5. In case of the 4-th order k-periodic recurrence fractions with the preperiod of l, all the considerations and relevant algorithms are similar.

For $l = 5$ and $k = 5$, the fourth-order mixed periodic recurrence fraction is written as

$$\left[\begin{array}{c|cccccccccccc}
q_0^* & & & & & & & & & & & & \\
\frac{p_1^*}{q_1^*} & q_1^* & & & & & & & & & & & \\
\frac{r_2^*}{p_2^*} & \frac{p_2^*}{q_2^*} & q_2^* & & & & & & & & & & \\
\frac{s_3^*}{r_3^*} & \frac{r_3^*}{p_3^*} & \frac{p_3^*}{q_3^*} & q_3^* & & & & & & & & & \\
0 & \frac{s_4^*}{r_4^*} & \frac{r_4^*}{p_4^*} & \frac{p_4^*}{q_4^*} & q_4^* & & & & & & & & \\
0 & 0 & \frac{s_1}{r_1} & \frac{r_1}{p_1} & \frac{p_1}{q_1} & q_1 & & & & & & & \\
0 & 0 & 0 & \frac{s_2}{r_2} & \frac{r_2}{p_2} & \frac{p_2}{q_2} & q_2 & & & & & & \\
0 & 0 & 0 & 0 & \frac{s_3}{r_3} & \frac{r_3}{p_3} & \frac{p_3}{q_3} & q_3 & & & & & \\
0 & 0 & 0 & 0 & 0 & \frac{s_4}{r_4} & \frac{r_4}{p_4} & \frac{p_4}{q_4} & q_4 & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{s_5}{r_5} & \frac{r_5}{p_5} & \frac{p_5}{q_5} & q_5 & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{s_1}{r_1} & \frac{r_1}{p_1} & \frac{p_1}{q_1} & q_1 & & \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots &
\end{array}\right]_{\infty},$$

where q_j^*, p_j^*, r_j^*, s_j^*, q_i, p_i, r_i and s_i are positive.

We shall decompose the numerator of the rational shortening $\frac{[q_0^*]_{n+5}}{[q_1^*]_{n+4}}$ of this fraction by the elements of the inscribed rectangular matrix $T(6)$, and the denominator — by the elements of the table $T(5)$. We shall obtain

$$\frac{[q_0^*]_{n+5}}{[q_1^*]_{n+4}} = \frac{[q_0^*]_5(q_1[q_2]_{n-1} + p_2[q_3]_{n-2} + r_3[q_4]_{n-3} + s_4[q_5]_{n-4}) +}{[q_1^*]_4(q_1[q_2]_{n-1} + p_2[q_3]_{n-2} + r_3[q_4]_{n-3} + s_4[q_5]_{n-4}) +}$$

$$\frac{+[q_0^*]_4(p_1[q_2]_{n-1} + r_2[q_3]_{n-2} + s_3[q_4]_{n-3}) + [q_0^*]_3(r_1[q_2]_{n-1} + s_2[q_3]_{n-2}) + [q_0^*]_2 s_1[q_2]_{n-1}}{+[q_1^*]_3(p_1[q_2]_{n-1} + r_2[q_3]_{n-2} + s_3[q_4]_{n-3}) + [q_1^*]_2(r_1[q_2]_{n-1} + s_2[q_3]_{n-2}) + [q_1^*]_1 s_1[q_2]_{n-1}}.$$

In the numerator and denominator, the expression in the first brackets is decomposition of the parapermanent $[q_1]_n$ by the elements of the first column, so the last fraction is written as

$$\frac{[q_0^*]_5[q_1]_n \ + \ [q_0^*]_4(p_1[q_2]_{n-1} + r_2[q_3]_{n-2} + s_3[q_4]_{n-3}) +}{[q_1^*]_4[q_1]_n \ + \ [q_1^*]_3(p_1[q_2]_{n-1} + r_2[q_3]_{n-2} + s_3[q_4]_{n-3}) +}$$

$$\frac{+[q_0^*]_3(r_1[q_2]_{n-1} + s_2[q_3]_{n-2}) + [q_0^*]_2 s_1[q_2]_{n-1}}{+[q_1^*]_2(r_1[q_2]_{n-1} + s_2[q_3]_{n-2}) + [q_1^*]_1 s_1[q_2]_{n-1}},$$

or after grouping corresponding summands, it is written as

$$\frac{[q_0^*]_5[q_1]_n + ([q_0^*]_4 p_1 + [q_0^*]_3 r_1 + [q_0^*]_2 s_1)[q_2]_{n-1} + ([q_0^*]_4 r_2 + [q_0^*]_3 s_2)[q_3]_{n-2} + [q_0^*]_4 s_3}{[q_1^*]_4[q_1]_n + ([q_1^*]_3 p_1 + [q_1^*]_2 r_1 + [q_1^*]_1 s_1)[q_2]_{n-1} + ([q_1^*]_3 r_2 + [q_1^*]_2 s_2)[q_3]_{n-2} + [q_1^*]_3 s_3}.$$

We shall divide the numerator and denominator of the last fraction by $[q_4]_{n-3}$ and obtain the fraction

$$\frac{[q_0^*]_5 \frac{[q_1]_n}{[q_2]_{n-1}} \frac{[q_2]_{n-1}}{[q_3]_{n-2}} \frac{[q_3]_{n-2}}{[q_4]_{n-3}} + ([q_0^*]_4 p_1 + [q_0^*]_3 r_1 + [q_0^*]_2 s_1) \frac{[q_2]_{n-1}}{[q_3]_{n-2}} \frac{[q_3]_{n-2}}{[q_4]_{n-3}} +}{[q_1^*]_4 \frac{[q_1]_n}{[q_2]_{n-1}} \frac{[q_2]_{n-1}}{[q_3]_{n-2}} \frac{[q_3]_{n-2}}{[q_4]_{n-3}} + ([q_1^*]_3 p_1 + [q_1^*]_2 r_1 + [q_1^*]_1 s_1) \frac{[q_2]_{n-1}}{[q_3]_{n-2}} \frac{[q_3]_{n-2}}{[q_4]_{n-3}} +}$$

$$\frac{+([q_0^*]_4 r_2+[q_0^*]_3 s_2)\frac{[q_3]_{n-2}}{[q_4]_{n-3}}+[q_0^*]_4 s_3}{+([q_1^*]_3 r_2+[q_1^*]_2 s_2)\frac{[q_3]_{n-2}}{[q_4]_{n-3}}+[q_1^*]_3 s_3}$$

Let us have the following limits

$$\lim_{n\to\infty}\frac{[q_0^*]_{n+4}}{[q_1^*]_{n+3}}=x^*,\ \lim_{n\to\infty}\frac{[q_1]_n}{[q_2]_{n-1}}=x,\ \lim_{n\to\infty}\frac{[q_2]_{n-1}}{[q_3]_{n-2}}=y,\ \lim_{n\to\infty}\frac{[q_3]_{n-2}}{[q_4]_{n-3}}=z.$$

Then the last expression is written as

$$x^*=\frac{[q_0^*]_5 xyz+([q_0^*]_4 p_1+[q_0^*]_3 r_1+[q_0^*]_2 s_1)yz+([q_0^*]_4 r_2+[q_0^*]_3 s_2)z+[q_0^*]_4 s_3}{[q_1^*]_4 xyz+([q_1^*]_3 p_1+[q_1^*]_2 r_1+[q_1^*]_1 s_1)yz+([q_1^*]_3 r_2+[q_1^*]_2 s_2)z+[q_1^*]_3 s_3},$$

where x, y, z are solutions of simultaneous equations

$$\begin{cases} x=q_1+\frac{p_2}{y}+\frac{r_3}{yz}+\frac{s_4}{yzu}, \\ y=q_2+\frac{p_3}{z}+\frac{r_4}{zu}+\frac{s_5}{zuv}, \\ z=q_3+\frac{p_4}{u}+\frac{r_5}{uv}+\frac{s_1}{uvx}, \\ u=q_4+\frac{p_5}{v}+\frac{r_1}{vx}+\frac{s_2}{vxy}, \\ v=q_5+\frac{p_1}{x}+\frac{r_2}{xy}+\frac{s_3}{xyz}, \end{cases} \tag{2.2}$$

for the 5-periodic recurrence fraction

$$\left[\begin{array}{c|cccccc} q_1 & & & & & & \\ \frac{p_2}{q_2} & q_2 & & & & & \\ \frac{r_3}{p_3} & \frac{p_3}{q_3} & q_3 & & & & \\ \frac{s_4}{r_4} & \frac{r_4}{p_4} & \frac{p_4}{q_4} & q_4 & & & \\ 0 & \frac{s_5}{r_5} & \frac{r_5}{p_5} & \frac{p_5}{q_5} & q_5 & & \\ 0 & 0 & \frac{s_1}{r_1} & \frac{r_1}{p_1} & \frac{p_1}{q_1} & q_1 & \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots \end{array}\right], \tag{2.3}$$

where

$$u=\lim_{n\to\infty}\frac{[q_4]_{n-3}}{[q_5]_{n-4}},\ v=\lim_{n\to\infty}\frac{[q_5]_{n-4}}{[q_1]_{n-5}}$$

(see [5], p.2 on page 13).

Thus, the theorem is true.

Theorem 2.2. *Let $l=5$, $k=5$, and $q_j^*, p_j^*, r_j^*, s_j^*, q_i, p_i, r_i, s_i>0$, and the limits*

$$lim_{n\to\infty}\frac{[q_0^*]_{n+4}}{[q_1^*]_{n+3}}=x^*,\ lim_{n\to\infty}\frac{[q_1]_n}{[q_2]_{n-1}}=x,\ lim_{n\to\infty}\frac{[q_2]_{n-1}}{[q_3]_{n-2}}=y,\ lim_{n\to\infty}\frac{[q_3]_{n-2}}{[q_4]_{n-3}}=z.$$

Then

$$x^*=\frac{[q_0^*]_5 xyz+([q_0^*]_4 p_1+[q_0^*]_3 r_1+[q_0^*]_2 s_1)yz+([q_0^*]_4 r_2+[q_0^*]_3 s_2)z+[q_0^*]_4 s_3}{[q_1^*]_4 xyz+([q_1^*]_3 p_1+[q_1^*]_2 r_1+[q_1^*]_1 s_1)yz+([q_1^*]_3 r_2+[q_1^*]_2 s_2)z+[q_1^*]_3 s_3},$$

where x, y, z are solutions of the simultaneous equations (2.2) for the 5-periodic recurrence fraction (2.3).

Example 2.3. Let

$$q_0^* = 3, q_1^* = 2, q_2^* = 3, q_3^* = 2, q_4^* = 3, p_1^* = 4, p_2^* = 3,$$
$$p_3^* = 4, p_4^* = 3, r_2^* = 2, r_3^* = 5, r_4^* = 2, s_3^* = 1, s_4^* = 1,$$
$$q_1 = 1, p_2 = 1, r_3 = 1, s_4 = 1, q_2 = 1, p_3 = 1, r_4 = 1, s_5 = 1, q_3 = 2, p_4 = 2,$$
$$r_5 = 2, s_1 = 2, q_4 = 1, p_5 = 1, r_1 = 1, s_2 = 1, q_5 = 2, p_1 = 2, r_2 = 2, s_3 = 2,$$

then the mixed recurrence fraction is written as

$$\left[\begin{array}{c|cccccccccc}
3 & & & & & & & & & & \\
\frac{4}{2} & 2 & & & & & & & & & \\
\frac{2}{3} & \frac{3}{3} & 3 & & & & & & & & \\
\frac{1}{5} & \frac{5}{4} & \frac{4}{2} & 2 & & & & & & & \\
0 & \frac{1}{2} & \frac{2}{3} & \frac{3}{3} & 3 & & & & & & \\
0 & 0 & \frac{2}{1} & \frac{1}{2} & \frac{2}{1} & 1 & & & & & \\
0 & 0 & 0 & \frac{1}{2} & \frac{2}{1} & \frac{1}{1} & 1 & & & & \\
0 & 0 & 0 & 0 & \frac{2}{1} & \frac{1}{1} & \frac{1}{2} & 2 & & & \\
0 & 0 & 0 & 0 & 0 & \frac{1}{1} & \frac{1}{2} & \frac{2}{1} & 1 & & \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{2}{1} & \frac{1}{2} & 2 & \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots
\end{array}\right]_{\infty}$$

The value of the mixed periodic recurrence fraction converges to the number

$$x^* = \frac{560xyz + 337yz + 317z + 276}{125xyz + 75yz + 71z + 62},$$

where x is the root of the equation modulo maximum

$$9x^4 - 9x^3 - 9x^2 - 8x - 16 = 0,$$

such that

$$x = \frac{1}{4} + \frac{1}{2}\left(\sqrt{-\frac{11}{8} + 2v} + \sqrt{\frac{33}{8} - 2v + \frac{\frac{109}{36}}{\sqrt{-\frac{11}{8} + 2v}}}\right) \approx 1.969558741906025,$$

where

$$v = \frac{1}{2}\left(\frac{55}{24} + \frac{1}{9}\sqrt[3]{-2007 + 144\sqrt{622}} - \frac{23}{\sqrt[3]{-2007 + 144\sqrt{622}}}\right),$$

and

$$y = \frac{4 - x}{3x^3 - 3x^2 - 3x - 4},$$
$$z = \frac{4 - x}{3x^2y - 3xy - 4y + x - 4}.$$

Thus,

$$x^* \approx 4.47989948650800763333.$$

Let us find rational shortenings of this fraction; we shall get

$$\delta_1 = \frac{10}{2} = 5, \delta_2 = \frac{41}{9} = 4,56, \delta_3 = \frac{138}{31} = 4,452, \delta_4 = \frac{560}{125} \approx 4,48, \delta_5 = \frac{897}{200} \approx 4,485,$$

$$\delta_6 = \frac{1774}{396} \approx 4,479798, \delta_7 = \frac{5281}{1179} \approx 4,47922, \delta_8 = \frac{10286}{2296} \approx 4,47997, \delta_9 = \frac{30298}{6763} \approx 4,47996,$$

$$\delta_{10} = \frac{59699}{13326} \approx 4,479889, \delta_{11} = \frac{115850}{25860} \approx 4,4798917, \delta_{12} = \frac{342269}{76401} \approx 4,479902,$$

$$\delta_{13} = \frac{663966}{148210} \approx 4,4799001, \delta_{14} = \frac{1961600}{437867} \approx 4,47989915, \delta_{15} = \frac{3863501}{862408} \approx 4,47989931,$$

$$\delta_{16} = \frac{7495302}{1673096} \approx 4,47989954, \delta_{17} = \frac{22143637}{4942887} \approx 4,47989950003,$$

$$\delta_{18} = \frac{42959342}{9589354} \approx 4,4798994802.$$

3. Algorithm for Calculation of Rational Shortenings of the Fourth-Order Mixed Periodic Recurrence Fraction

Let us construct the algorithm for calculation of rational shortenings of the fourth-order mixed periodic recurrence fractions, which is much more practical than the algorithm described in the previous section.

Let n be the order of the parameter of its rational shortening, and $n = sk + l,\ s = 1, 2, 3, \ldots$. Then the following theorem is true.

Theorem 3.1. *The rational shortening*

$$\delta_n^* = \frac{P_n^*}{Q_n^*}$$

of the fourth-order mixed periodic recurrence fraction (2.1), with the period of $k \geqslant 2$, equals the value of the expression

$$q_0^* + p_1^* \cdot \frac{B}{A} + r_2^* \cdot \frac{C}{A} + s_3^* \cdot \frac{D}{A},$$

where A , B , C and D are defined by the recurrence equalities

$$A = s_3\alpha_{l-1}D_{ks-3}^{s-1} + (s_2\alpha_{l-2} + r_2\alpha_{l-1})C_{ks-2}^{s-1} + (s_1\alpha_{l-3} + r_1\alpha_{l-2} + p_1\alpha_{l-1})B_{ks-1}^{s-1} + \alpha_l A_{ks}^s, \quad (3.1)$$

$$B = s_3\beta_{l-2}D_{ks-3}^{s-1} + (s_2\beta_{l-3} + r_2\beta_{l-2})C_{ks-2}^{s-1} + (s_1\beta_{l-4} + r_1\beta_{l-3} + p_1\beta_{l-2})B_{ks-1}^{s-1} + \beta_{l-1} A_{ks}^s, \quad (3.2)$$

$$C = s_3\gamma_{l-3}D_{ks-3}^{s-1} + (s_2\gamma_{l-4} + r_2\gamma_{l-3})C_{ks-2}^{s-1} + (s_1\gamma_{l-5} + r_1\gamma_{l-4} + p_1\gamma_{l-3})B_{ks-1}^{s-1} + \gamma_{l-2} A_{ks}^s, \quad (3.3)$$

$$D = s_3\eta_{l-4}D_{ks-3}^{s-1} + (s_2\eta_{l-5} + r_2\eta_{l-4})C_{ks-2}^{s-1} + (s_1\eta_{l-6} + r_1\eta_{l-5} + p_1\eta_{l-4})B_{ks-1}^{s-1} + \eta_{l-3} A_{ks}^s, \quad (3.4)$$

where

$$\alpha_l = \begin{bmatrix} q_1^* & & & & & & & & & \\ \frac{p_2^*}{q_2^*} & q_2^* & & & & & & & & \\ \frac{r_3^*}{p_3^*} & \frac{p_3^*}{q_3^*} & q_3^* & & & & & & & \\ \frac{s_4^*}{r_4^*} & \frac{r_4^*}{p_4^*} & \frac{p_4^*}{q_4^*} & q_{4^*} & & & & & & \\ 0 & \frac{s_5^*}{r_5^*} & \frac{r_5^*}{p_5^*} & \frac{p_5^*}{q_5^*} & q_{5^*} & & & & & \\ \vdots & \ldots & \ldots & \ldots & \ldots & \ddots & & & & \\ 0 & 0 & 0 & 0 & 0 & \ldots & q_{l-3}^* & & & \\ 0 & 0 & 0 & 0 & 0 & \ldots & \frac{p_{l-2}^*}{q_{l-2}^*} & q_{l-2}^* & & \\ 0 & 0 & 0 & 0 & 0 & \ldots & \frac{r_{l-1}^*}{p_{l-1}^*} & \frac{p_{l-1}^*}{q_{l-1}^*} & q_{l-1}^* & \\ 0 & 0 & 0 & 0 & 0 & \ldots & \frac{s_l^*}{r_l^*} & \frac{r_l^*}{p_l^*} & \frac{p_l^*}{q_l^*} & q_l^* \end{bmatrix}, \tag{3.5}$$

$$\beta_{l-1} = \begin{bmatrix} q_2^* & & & & & & & & & \\ \frac{p_3^*}{q_3^*} & q_3^* & & & & & & & & \\ \frac{r_4^*}{p_4^*} & \frac{p_4^*}{q_4^*} & q_4^* & & & & & & & \\ \frac{s_5^*}{r_5^*} & \frac{r_5^*}{p_5^*} & \frac{p_5^*}{q_5^*} & q_{5^*} & & & & & & \\ 0 & \frac{s_6^*}{r_6^*} & \frac{r_6^*}{p_6^*} & \frac{p_6^*}{q_6^*} & q_{6^*} & & & & & \\ \vdots & \ldots & \ldots & \ldots & \ldots & \ddots & & & & \\ 0 & 0 & 0 & 0 & 0 & \ldots & q_{l-3}^* & & & \\ 0 & 0 & 0 & 0 & 0 & \ldots & \frac{p_{l-2}^*}{q_{l-2}^*} & q_{l-2}^* & & \\ 0 & 0 & 0 & 0 & 0 & \ldots & \frac{r_{l-1}^*}{p_{l-1}^*} & \frac{p_{l-1}^*}{q_{l-1}^*} & q_{l-1}^* & \\ 0 & 0 & 0 & 0 & 0 & \ldots & \frac{s_l^*}{r_l^*} & \frac{r_l^*}{p_l^*} & \frac{p_l^*}{q_l^*} & q_l^* \end{bmatrix}, \tag{3.6}$$

$$\gamma_{l-2} = \begin{bmatrix} q_3^* & & & & & & & & & \\ \frac{p_4^*}{q_4^*} & q_4^* & & & & & & & & \\ \frac{r_5^*}{p_5^*} & \frac{p_5^*}{q_5^*} & q_5^* & & & & & & & \\ \frac{s_6^*}{r_6^*} & \frac{r_6^*}{p_6^*} & \frac{p_6^*}{q_6^*} & q_{6^*} & & & & & & \\ 0 & \frac{s_7^*}{r_7^*} & \frac{r_7^*}{p_7^*} & \frac{p_7^*}{q_7^*} & q_{7^*} & & & & & \\ \vdots & \ldots & \ldots & \ldots & \ldots & \ddots & & & & \\ 0 & 0 & 0 & 0 & 0 & \ldots & q_{l-3}^* & & & \\ 0 & 0 & 0 & 0 & 0 & \ldots & \frac{p_{l-2}^*}{q_{l-2}^*} & q_{l-2}^* & & \\ 0 & 0 & 0 & 0 & 0 & \ldots & \frac{r_{l-1}^*}{p_{l-1}^*} & \frac{p_{l-1}^*}{q_{l-1}^*} & q_{l-1}^* & \\ 0 & 0 & 0 & 0 & 0 & \ldots & \frac{s_l^*}{r_l^*} & \frac{r_l^*}{p_l^*} & \frac{p_l^*}{q_l^*} & q_l^* \end{bmatrix}, \tag{3.7}$$

$$
\eta_{l-3}=\left[\begin{array}{cccccccccc}
q_4^* \\
\frac{p_5^*}{q_5^*} & q_5^* \\
\frac{r_6^*}{p_6^*} & \frac{p_6^*}{q_6^*} & q_6^* \\
\frac{s_7^*}{r_7^*} & \frac{r_7^*}{p_7^*} & \frac{p_7^*}{q_7^*} & q_{7^*} \\
0 & \frac{s_8^*}{r_8^*} & \frac{r_8^*}{p_8^*} & \frac{p_8^*}{q_8^*} & q_{8^*} \\
\vdots & \dots & \dots & \dots & \dots & \ddots \\
0 & 0 & 0 & 0 & 0 & \dots & q_{l-3}^* \\
0 & 0 & 0 & 0 & 0 & \dots & \frac{p_{l-2}^*}{q_{l-2}^*} & q_{l-2}^* \\
0 & 0 & 0 & 0 & 0 & \dots & \frac{r_{l-1}^*}{p_{l-1}^*} & \frac{p_{l-1}^*}{q_{l-1}^*} & q_{l-1}^* \\
0 & 0 & 0 & 0 & 0 & \dots & \frac{s_l^*}{r_l^*} & \frac{r_l^*}{p_l^*} & \frac{p_l^*}{q_l^*} & q_l^*
\end{array}\right], \tag{3.8}
$$

and A_{sk}^{s}, B_{sk-1}^{s-1}, C_{sk-2}^{s-1}, D_{sk-3}^{s-1} *are respectively defined by the recurrences (9'), (10'), (11'), (12'). If* $k=2,3,4$, *we believe* $\xi_{<0}=\tau_{<0}=\psi_{<0}=\varphi_{<0}=0$, $\varphi_0=\psi_0=\tau_0=\xi_0=1$. *Likewise, if* $l=2,3,4$, *we consider* $\alpha_{<0}=\beta_{<0}=\gamma_{<0}=\eta_{<0}=0$, $\quad \alpha_0=\beta_0=\gamma_0=\eta_0=1$.

Proof. For the fraction (2.1), the parapermanents P_{sk}^* and Q_{sk}^* are written as

$$
P_{sk}^*=\left[\begin{array}{ccccccccccccccccccc}
q_0^* \\
\frac{p_1^*}{q_1^*} & q_1^* \\
\frac{r_2^*}{p_2^*} & \frac{p_2^*}{q_2^*} & q_2^* \\
\frac{s_3^*}{r_3^*} & \frac{r_3^*}{p_3^*} & \frac{p_3^*}{q_3^*} & q_3^* \\
0 & \frac{s_4^*}{r_4^*} & \frac{r_4^*}{p_4^*} & \frac{p_4^*}{q_4^*} & q_4^* \\
\vdots & \dots & \dots & \dots & \dots & \ddots \\
0 & 0 & 0 & 0 & 0 & \dots & q_{l-3}^* \\
0 & 0 & 0 & 0 & 0 & \dots & \frac{p_{l-2}^*}{q_{l-2}^*} & q_{l-2}^* \\
0 & 0 & 0 & 0 & 0 & \dots & \frac{r_{l-1}^*}{p_{l-1}^*} & \frac{p_{l-1}^*}{q_{l-1}^*} & q_{l-1}^* \\
0 & 0 & 0 & 0 & 0 & \dots & \frac{s_l^*}{r_l^*} & \frac{r_l^*}{p_l^*} & \frac{p_l^*}{q_l^*} & q_l^* \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{s_1}{r_1} & \frac{r_1}{p_1} & \frac{p_1}{q_1} & q_1 \\
\vdots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \ddots \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & q_{k-3} \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & \frac{p_{k-2}}{q_{k-2}} & q_{k-2} \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & \frac{r_{k-1}}{p_{k-1}} & \frac{p_{k-1}}{q_{k-1}} & q_{k-1} \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & \frac{s_k}{r_k} & \frac{r_k}{p_k} & \frac{p_k}{q_k} & q_k \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{s_1}{r_1} & \frac{r_1}{p_1} & \frac{p_1}{q_1} & q_1 \\
\vdots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \ddots \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & q_k
\end{array}\right], \tag{3.9}
$$

$$
Q_{sk}^{*}=\left[\begin{array}{ccccccccccccccccccc}
q_1^{*} &&&&&&&&&&&&&&&&&& \\
\frac{p_2^{*}}{q_2^{*}} & q_2^{*} &&&&&&&&&&&&&&&&& \\
\frac{r_3^{*}}{p_3^{*}} & \frac{p_3^{*}}{q_3^{*}} & q_3^{*} &&&&&&&&&&&&&&&& \\
\frac{s_4^{*}}{r_4^{*}} & \frac{r_4^{*}}{p_4^{*}} & \frac{p_4^{*}}{q_4^{*}} & q_4^{*} &&&&&&&&&&&&&&& \\
0 & \frac{s_5^{*}}{r_5^{*}} & \frac{r_5^{*}}{p_5^{*}} & \frac{p_5^{*}}{q_5^{*}} & q_5^{*} &&&&&&&&&&&&&& \\
\vdots & \ldots & \ldots & \ldots & \ldots & \ddots &&&&&&&&&&&&& \\
0 & 0 & 0 & 0 & 0 & \ldots & q_{l-3}^{*} &&&&&&&&&&&& \\
0 & 0 & 0 & 0 & 0 & \ldots & \frac{p_{l-2}^{*}}{q_{l-2}^{*}} & q_{l-2}^{*} &&&&&&&&&&& \\
0 & 0 & 0 & 0 & 0 & \ldots & \frac{r_{l-1}^{*}}{p_{l-1}^{*}} & \frac{p_{l-1}^{*}}{q_{l-1}^{*}} & q_{l-1}^{*} &&&&&&&&&& \\
0 & 0 & 0 & 0 & 0 & \ldots & \frac{s_l^{*}}{r_l^{*}} & \frac{r_l^{*}}{p_l^{*}} & \frac{p_l^{*}}{q_l^{*}} & q_l^{*} &&&&&&&&& \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & \frac{s_1}{r_1} & \frac{r_1}{p_1} & \frac{p_1}{q_1} & q_1 &&&&&&&& \\
\vdots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ddots &&&&&&& \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & \ldots & q_{k-3} &&&&&& \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & \ldots & \frac{p_{k-2}}{q_{k-2}} & q_{k-2} &&&&& \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & \ldots & \frac{r_{k-1}}{p_{k-1}} & \frac{p_{k-1}}{q_{k-1}} & q_{k-1} &&&& \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & \ldots & \frac{s_k}{r_k} & \frac{r_k}{p_k} & \frac{p_k}{q_k} & q_k &&& \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & \frac{s_1}{r_1} & \frac{r_1}{p_1} & \frac{p_1}{q_1} & q_1 && \\
\vdots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ddots & \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & \ldots & q_k
\end{array}\right].
$$

We shall denote the parapermanent, derived from the parapermanent (3.9) as a result of deleting the first column, by A, the parapermanent, derived as a result of deleting the first two columns, — by B, the parapermanent, derived as a result of deleting the first three columns, — by C, and the parapermanent, derived as a result of deleting the first four columns, — by D. Then, decomposing the first parapermanent by the elements of the first column, we shall get the equality $P_{sk}^{*}=q_0^{*}A+p_1^{*}B+r_2^{*}C+s_3^{*}D$, which, by taking the equality $Q_{sk}^{*}=A$ into account, will lead to the equality $\delta_n^{*}=q_0^{*}+p_1^{*}\frac{B}{A}+r_2^{*}\frac{C}{A}+s_3^{*}\frac{D}{A}$.

We shall denote the parapermanent, derived from the parapermanent (3.9) as a result of deleting the first $(l+1)$ columns, by A_{sk}^{s} (it contains s periods of the fraction). The parapermanent, derived from the parapermanent A_{sk}^{s} as a result of deleting the first column, is denoted by B_{sk-1}^{s-1}. The parapermanent, derived from the parapermanent A_{sk}^{s} as a result of deleting the first two columns, is denoted by C_{sk-2}^{s-1}, and the parapermanent, derived from the parapermanent A_{sk}^{s} as a result of deleting the first three columns, — by D_{sk-3}^{s-1}. We shall decompose the parapermanent A by the elements of the inscribed rectangular matrix $T(l+1)$. Taking (3.5) into account and applying the theorem on incomplete decomposition of a parapermanent by the elements of the $(l+1)$-th column, we obtain the equality (3.1). We shall make similar transformations with the parapermanent B. We decompose it by the elements of the table $T(l)$, taking (3.6) into account, at that we get the equality (3.2). We shall also make similar transformations with the parapermanents C, D. We decompose them by the elements of the tables $T(l-1), T(l-2)$ respectively and take (3.7), (3.8) into account. At that, we get the equalities (3.3), (3.4).

We shall decompose the parapermanent A^s_{sk} by the elements of the inscribed rectangular table $T(k+1)$, then we get the recurrence (9′). Likewise, we shall deal with the parapermanents B^{s-1}_{sk-1}, C^{s-1}_{sk-2} and D^{s-1}_{sk-3}, decomposing them by the elements of the tables $T(k)$, $T(k-1)$ and $T(k-2)$ respectively. At that, we get the recurrences (10′), (11′), (12′). $\square$

Example 3.2. Let us have the fourth-order mixed 5-periodic recurrence fraction from Example 2.3.

Let us find its rational shortenings applying Theorem *3.1*.

We shall find

$$\eta_{-2}, \eta_{-1}, \eta_0, \eta_1, \gamma_{-1}, \gamma_0, \gamma_1, \gamma_2, \beta_0, \beta_1, \beta_2, \beta_3, \alpha_1, \alpha_2, \alpha_3, \alpha_4,$$

$$\xi_{-1}, \xi_0, \xi_1, \xi_2, \tau_0, \tau_1, \tau_2, \tau_3, \psi_1, \psi_2, \psi_3, \psi_4, \varphi_2, \varphi_3, \varphi_4, \varphi_5$$

from the equalities *(3.5), (3.6), (3.7), (3.8), (13′), (14′), (15′), (16′)*:

$$\eta_{-2} = 0, \eta_{-1} = 0, \eta_0 = 1, \eta_1 = 3, \gamma_{-1} = 0, \gamma_0 = 1, \gamma_1 = 2, \gamma_2 = \left[\begin{array}{ll} 2 & \\ \frac{3}{3} & 3 \end{array}\right] = 9,$$

$$\beta_0 = 1, \quad \beta_1 = 3, \quad \beta_2 = \left[\begin{array}{ll} 3 & \\ \frac{4}{2} & 2 \end{array}\right] = 10, \quad \beta_3 = \left[\begin{array}{lll} 3 & & \\ \frac{4}{2} & 2 & \\ \frac{2}{3} & \frac{3}{3} & 3 \end{array}\right] = 41,$$

$$\alpha_1 = 2,\ \alpha_2 = \left[\begin{array}{ll} 2 & \\ \frac{3}{3} & 3 \end{array}\right] = 9,\ \alpha_3 = \left[\begin{array}{lll} 2 & & \\ \frac{3}{3} & 3 & \\ \frac{5}{4} & \frac{4}{2} & 2 \end{array}\right] = 31,\ \alpha_4 = \left[\begin{array}{llll} 2 & & & \\ \frac{3}{3} & 3 & & \\ \frac{5}{4} & \frac{4}{2} & 2 & \\ \frac{1}{2} & \frac{1}{3} & \frac{3}{3} & 3 \end{array}\right] = 125,$$

$$\xi_{-1} = 0, \quad \xi_0 = 1, \quad \xi_1 = 1, \quad \xi_2 = \left[\begin{array}{ll} 1 & \\ \frac{1}{2} & 2 \end{array}\right] = 4,$$

$$\tau_0 = 1, \quad \tau_1 = 2, \quad \tau_2 = \left[\begin{array}{ll} 2 & \\ \frac{2}{1} & 1 \end{array}\right] = 4, \quad \tau_3 = \left[\begin{array}{lll} 2 & & \\ \frac{2}{1} & 1 & \\ \frac{2}{1} & \frac{1}{2} & 2 \end{array}\right] = 12,$$

$$\psi_1 = 1, \quad \psi_2 = \left[\begin{array}{ll} 1 & \\ \frac{1}{2} & 2 \end{array}\right] = 3, \quad \psi_3 = \left[\begin{array}{lll} 1 & & \\ \frac{1}{2} & 2 & \\ \frac{1}{2} & \frac{2}{1} & 1 \end{array}\right] = 6,$$

$$\psi_4 = \left[\begin{array}{llll} 1 & & & \\ \frac{1}{2} & 2 & & \\ \frac{1}{2} & \frac{2}{1} & 1 & \\ \frac{1}{2} & \frac{2}{1} & \frac{1}{2} & 2 \end{array}\right] = 11, \quad \varphi_2 = \left[\begin{array}{ll} 1 & \\ \frac{1}{1} & 1 \end{array}\right] = 2, \quad \varphi_3 = \left[\begin{array}{lll} 1 & & \\ \frac{1}{1} & 1 & \\ \frac{1}{1} & \frac{1}{2} & 2 \end{array}\right] = 6,$$

$$\varphi_4 = \left[\begin{array}{llll} 1 & & & \\ \frac{1}{1} & 1 & & \\ \frac{1}{1} & \frac{1}{2} & 2 & \\ \frac{1}{1} & \frac{1}{2} & \frac{2}{1} & 1 \end{array}\right] = 12, \quad \varphi_5 = \left[\begin{array}{lllll} 1 & & & & \\ \frac{1}{1} & 1 & & & \\ \frac{1}{1} & \frac{1}{2} & 2 & & \\ \frac{1}{1} & \frac{1}{2} & \frac{2}{1} & 1 & \\ 0 & \frac{1}{2} & \frac{2}{1} & \frac{1}{2} & 2 \end{array}\right] = 35.$$

Thus, the recurrences (3.1), (3.2), (3.3), (3.4), (9′), (10′), (11′), (12′) will be written respectively as

$$A = 62D^{s-1}_{5s-3} + 71C^{s-1}_{5s-2} + 75B^{s-1}_{5s-1} + 125A^s_{5s},$$

$$B = 20D^{s-1}_{5s-3} + 23C^{s-1}_{5s-2} + 25B^{s-1}_{5s-1} + 41A^{s}_{5s},$$

$$C = 4D^{s-1}_{5s-3} + 5C^{s-1}_{5s-2} + 5B^{s-1}_{5s-1} + 9A^{s}_{5s},$$

$$D = 2D^{s-1}_{5s-3} + 2C^{s-1}_{5s-2} + 2B^{s-1}_{5s-1} + 3A^{s}_{5s},$$

$$A^{s}_{5s} = 24D^{s-2}_{5s-8} + 30C^{s-2}_{5s-7} + 34B^{s-2}_{5s-6} + 35A^{s-1}_{5s-5},$$

$$B^{s-1}_{5s-1} = 12D^{s-2}_{5s-8} + 15C^{s-2}_{5s-7} + 17B^{s-2}_{5s-6} + 18A^{s-1}_{5s-5},$$

$$C^{s-1}_{5s-2} = 8D^{s-2}_{5s-8} + 10C^{s-2}_{5s-7} + 12B^{s-2}_{5s-6} + 12A^{s-1}_{5s-5},$$

$$D^{s-1}_{5s-3} = 2D^{s-2}_{5s-8} + 3C^{s-2}_{5s-7} + 3B^{s-2}_{5s-6} + 3A^{s-1}_{5s-5}.$$

s-th convergence to the value of the given recurrence fraction by the algorithm of Theorem 3.1 is written as

$$\gamma_s = 3 + 4\frac{B}{A} + 2\frac{C}{A} + \frac{D}{A}.$$

Since

$$D^0_2 = \xi_2 = 3, \quad C^0_3 = \tau_3 = 12, \quad B^0_4 = \psi_4 = 18, \quad A^1_5 = \varphi_5 = 35,$$

then

$$\gamma_1 = \frac{59699}{13326} \approx 4,479889, \ \gamma_2 = \frac{3863501}{862408} \approx 4,47989931, \ \gamma_3 = \frac{1780047}{658189} \approx 2.70446179,$$

$$\gamma_4 = \frac{98034217}{36249057} \approx 2.70446180035, \ \gamma_5 = \frac{5399131407}{1996379245} \approx 2.70446180029286,$$

$$\gamma_6 = \frac{297351484441}{109948487499} \approx 2.704461800292654.$$

Thus, s-th convergence γ_s, found with the help of the algorithm of Theorem 3.1 coincides with the $(5s + l)$–th convergence δ_{5s+l}, found with the help of established recurrences.

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У роботі вивчаються змішані періодичні рекурентні дроби четвертого порядку. Запропоновано алгоритми їх обчислення і встановлено звязки з відповідними алгебраїчними рівняннями четвертого порядку.

Ключові слова: парафункції трикутних матриць, рекурентні дроби, алгебраїчні рівняння.

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LINEAR SUBSPACES IN ZEROS OF POLYNOMIALS ON BANACH SPACES

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Abstract. A survey of general results about linear subspaces in zeros of polynomials on real and complex Banach spaces.

Keywords: polynomials, linear subspaces, zeros of polynomials on Banach spaces.

1. INTRODUCTION

The paper is a survey of results related to linear subspaces in zero-sets (kernels) of real and complex polynomials on Banach spaces.

The study of the zeros of polynomials has a long history, which began with the results obtained in algebraic geometry and complex analysis. Zeros of polynomials on infinite demensional Banach spaces was studied in [1], [2], [3], [4], [5], [7], [12], [13], [18] by R. Aron, C. Boyd, R. Ryan, I. Zalduendo, D. Garsia, M. Maestre, A. Zagorodnyuk, A. Plichko, R. Gonzalo, J. Ferrer, P. Hajek and others.

Let X and Y be real or complex Banach vector spaces. For every positive integer numbers $n, m \in \mathbb{N}$ let $X^n Y^m$ will denote the Cartesian product of n copies of X and m copies of Y, and $x^n y^m$ will denote the element $(x, \ldots, x, y, \ldots, y)$ from $X^n Y^m$.

For $n \in \mathbb{N}$ we denote by $\mathcal{L}(^n X, Y)$ the vector space of all continuous n-linear mappings F from X to Y endowed with the norm of uniform convergence on the unit ball of X^n. An n-linear mapping F is called symmetric if

$$F(x_1, \ldots, x_n) = F\left(x_{s(1)}, \ldots, x_{s(n)}\right), \qquad s \in \mathfrak{S}_n,$$

where $\mathfrak{S}_n$ means all permutations

$$s\colon \{1, \ldots, n\} \longmapsto \{s(1), \ldots, s(n)\}.$$

The subspace in $\mathcal{L}({}^nX,Y)$ of all continuous symmetric n-linear maps will be denoted by $\mathcal{L}_s({}^nX,Y)$. Clearly, $\mathcal{L}({}^nX,Y)$ and $\mathcal{L}_s({}^nX,Y)$ are Banach spaces. Further we will not write the index $n = 1$. In particular, $\mathcal{L}(X)$ denotes the algebra of all continuous linear operators and $\mathcal{L}(X,\mathbb{C}) := X'$ denotes the dual space of X.

Definition 1. Let us denote by Δ_n the natural embeddings called *diagonal mappings* from X to X^n defined as

$$\begin{aligned}\Delta_n\colon X &\longrightarrow X^n\\ x &\longmapsto (x,\ldots,x).\end{aligned}$$

A mapping P from X to Y is called a continuous n-homogeneous polynomial if

$$P(x) = (F\circ\Delta_n)(x) \quad \text{for some} \quad F\in\mathcal{L}({}^nX,Y). \tag{1.1}$$

Let $\mathcal{P}({}^nX,Y)$ denote the vector space of all continuous n-homogeneous polynomials endowed with the norm of uniform convergence on the unit ball B of X, i.e.,

$$\|P\| = \sup_{x\in B}\|P(x)\|$$

with $P\in\mathcal{P}({}^nX,Y)$.

In the paper we consider cases $Y=\mathbb{R}$ or $Y=\mathbb{C}$ the fields of real or complex numbers. We use notation $\mathcal{P}(X)$ and $\mathcal{P}({}^nX)$ for the space of scalar valued polynomials and n-homogeneous scalar valued polynomials respectively.

Let us denote by $\check{P}$ the unique symmetric n-linear map F which satisfies 1.1 for a given $P\in\mathcal{P}({}^nX)$.

For detales on polynomials on Banach spaces we refer the reader to [9], [10], [17].

2. Linear Subspaces in Zeros of Complex Polynomials

If X is an arbitrary complex vector space (not necessarily normed), we define a n-homogeneous complex polynomial by the formula

$$P(x) = (F\circ\Delta_n)(x) \qquad x\in X,$$

where F is a complex n-linear (not necessarily continuous) functional on X.

It is clear that the kernel (i.e. the set of zeros) of an n-homogeneous complex polynomial P on X, where $n>0$ and $\dim X>1$, consists of one-dimensional subspaces. In [18] A. Plichko and A. Zagorodnyuk showed that one-dimensional subspaces consists of infinite-dimensional subspaces if $\dim X=\infty$.

Theorem 2.1. *(*[18]*) Let X be an infinite-dimensional complex vector space and P is a complex n-homogeneous polynomial on X. Then there exists an infinite-dimensional subspace X_0 such that*

$$X_0\subset\ker P.$$

Lemma 1. *Let Theorem 2.1 be proved for every homogeneous polynomial of degree $\le n$. Then for arbitrary homogeneous polynomials $P_1,\cdots,P_m$ of degree $\le n$ there exists a subspace*

$$X_0\subset\ker P_1\cap\ldots\cap\ker P_m$$

such that $\dim X_0=\infty$.

Proof. Let $X_1 \subset \ker P_1$ with $\dim X_1 = \infty$. Then there exists a subspace $X_2 \subset X_1 \cap \ker P_2$ such that $\dim X_2 = \infty$. Continuing this process, we get the subspace

$$X_0 = X_m \subset X_{m-1} \subset \cdots \subset X_1$$

with $X_0 \subset \ker P_1 \cap \cdots \cap \ker P_m$ and $\dim X_0 = \infty$. □

Proof of Theorem 2.1. We will construct X_0 using the induction on n. Evidently that the theorem is true for linear functionals. Suppose that it is true for homogeneous polynomials of degree $< n$.

Let $x_1 \in X$ is chosen such that $P(x_1) \neq 0$ (if such x_1 does not exist then the assertion of theorems is true automatically). By the induction hypothesis and by Lemma 1 there exists a subspace $X_1 \subset X$ with $\dim X_1 = \infty$, on which each of the homogeneous polynomials

$$\begin{aligned} &P_{x_1}(x) := \check{P}\left(x_1, x^{n-1}\right), \\ &P_{x_1^2}(x) := \check{P}\left(x_1^2, x^{n-2}\right), \\ &\dots\dots\dots\dots\dots\dots\dots \\ &P_{x_1^{n-1}}(x) := \check{P}\left(x_1^{n-1}, x\right) \end{aligned}$$

vanish for all $x \in X_1$, where $\check{P}$ is the symmetric n-linear functional associated with the n-homogeneous polynomial P.

On second step we choose an element $x_2 \in X_1$ such that $P(x_2) \neq 0$ (if x_2 does not exist then $X_1 \subset \ker P$ and the theorem is proved at once). By the induction hypothesis and by Lemma 1 there exists a subspace $X_2 \subset X_1$ with $\dim X_2 = \infty$ on which each homogeneous polynomials

$$P_{x_1^k, x_2^l}(x) := \check{P}\left(x_1^k, x_2^l, x^{n-k-l}\right), \qquad 0 < k + l < n$$

vanish for all $x \in X_2$.

We continue this process in the way written above. If it finishes on the i-th step (i.e. $P(X_i) \equiv 0$), then the theorem is proved. If it does not finish then we will get an infinite sequence (x_i) consisting of linearly independent terms such that $P(x_i) \neq 0$ for every $i \in \mathbb{N}$ and

$$\check{P}\left(x_1^{k_1}, x_2^{k_2}, \dots, x_i^{k_i}\right) = 0$$

if $0 < k_i < n$ at least for one k_i.

Consequently, it follows that for any finite sequence of scalars (a_i),

$$P\left(\sum_i a_i x_i\right) = \sum_i a_i^n P(x_i).$$

Put $y_i = x_i / P(x_i)$ for all $i \in \mathbb{N}$. Then P vanishes on the linear span of elements

$$y_1 + \sqrt[n]{-1} y_2, \;\; y_3 + \sqrt[n]{-1} y_4, \;\; y_5 + \sqrt[n]{-1} y_6, \;\; \dots .$$

The theorem is proved. □

In [18] it was proved the following:

Corollary 1. *For every polynomial functional P on a complex infinite dimensional vector space, for which $P(0) = 0$, there exists an infinite dimensional linear subspace X_0 such that $X_0 \subset \ker P$.*

Corollary 2. *If P is a polynomial functional on a complex infinite dimensional vector space and $P(x_0) = 0$, then there exists an infinite dimensional affine subspace $X_0 \subset \ker P$ with $x_0 \in X_0$.*

In [19] it was proved next corollary:

Corollary 3. *There is a function $\Phi : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$, $\Phi(m,d) = n$ with the following property.*

For every complex polynomial $P : \mathbb{C}^n \to \mathbb{C}$ of degree d, there is a subspace $X \subset \mathbb{C}^n$ dimension m such that $P|_X \equiv P(0)$.

The real analogue of this result is obviously false, as can be seen by considering $P(x) = \sum x_j^2$. Despite this, a number of positive results hold. For example, one can show:

Theorem 2.2. *There is a function $\theta : \mathbb{N}_0 \to \mathbb{N}_0, \theta(m) = n$ with the following property:*

For every real polynomial $P : \mathbb{R}^n \to \mathbb{R}$ which is homogeneous of degree 3, there is a subspace $X \subset \mathbb{R}^n$ of dimension m such that $P|_X \equiv 0$.

Theorem 2.3. *If a real infinite dimensional Banach space E does not admit a 2-homogeneous positive definite polynomial, then every 2-homogeneous polynomial $P : E \to \mathbb{R}$ is identically 0 on an infinite dimensional subspace of E.*

3. Nonseparable Zero Subspaces

3.1. Nonseparable Subspaces in $\ker P \subset l_\infty$

All results of this subsection was proved in [11] by M. Fernandez-Unzueta. In particular in [11] was proved that every complex polynomial P defined on l_∞ such that $P(0) = 0$ necessarily vanishes on a non-separable subspace. In the real case, it was shown that if P vanishes on a copy of c_0, then it vanishes as well on a non-separable subspace.

Theorem 3.1. *Consider the Banach space l_∞ (real or complex) and a subspace $G \subset l_\infty$ isomorphic to c_0. Then, there exists a non-countable collection of vectors $(x_\alpha)_{\alpha \in A} \subset l_\infty$ satisfying the following condition: For every $\{P_i\}_{i=1}^\infty \subset \mathcal{P}(l_\infty)$ such that $G \subset \bigcap_{i=n}^\infty \ker P_i$, there exists a subset of indices $\Gamma \subset A$ with $A \setminus \Gamma$ at most countable such that the subspace*

$$F_\Gamma := \overline{Span\{x_\alpha; \alpha \in \Gamma\}}$$

is non-separable and contained in $\bigcap_{i=1}^\infty \ker P_i$.

Lemma 2. *It is enough to prove Theorem 3.1 for the case where the collection of polynomials $\{P_i\} \subset \mathcal{P}(^n l_\infty)$ reduces to a single homogeneous polynomial $P \in \mathcal{P}(^n l_\infty)$.*

Proof of Theorem 3.1. By Lemma 2 it is enough to consider the case of a single homogeneous polynomial P. The proof will be done by induction on n, the degree of the polynomial. At each inductive step n we will, however, assume that the result holds for a countable family of polynomials of degree strictly less than n. The case $n = 1$ asserts that for a fixed linear functional $x^* \in l_\infty$ such that $x^*|_{c_0} = 0$, there exists $\Gamma \subset A$, a subset of indices with countable complement in A, such that $x^*(x_\gamma) = 0$ if $\gamma \in \Gamma$.

We assume now that Theorem 3.1 holds for polynomials of degree $k < n$.

Let $P \in \mathcal{P}(^n l_\infty)$ be such that $c_0 \subset \ker P$ and let $(e_i)_i$ be the canonical basis of c_0. Consider the countable family of polynomials $P_{i_1,\ldots,i_k} \subset \mathcal{P}(^{n-k} l_\infty)$ defined as follows:

$$P_{i_1,\ldots,i_k}(x) := \check{P}(e_{i_k}, \ldots, e_{i_k}, x, \overset{n-k}{\ldots}, x) \text{ for } x \in l_\infty, \quad 1 \le k < n \text{ and } i_j \in \mathbb{N}. \tag{3.1}$$

Every polynomial in this countable collection has degree strictly less than n and satisfies $c_0 \subset \ker P_{i_1,\dots i_k}$. The induction hypothesis allows us to choose some set of indices

$$\Gamma_1 \subset A \tag{3.2}$$

in such a way that $A \supset \Gamma_1$ is countable and $F_{\Gamma_1} = \overline{\text{Span}\{x_\gamma; \gamma \in \Gamma_1\}}$ is a non-separable subspace contained in $\bigcap \ker P_{i_1,\dots i_k}$. A main step in the proof is the fact that the following set:

$$S = \{\gamma \in \Gamma_1;\ \text{there are } \alpha_2, \dots, \alpha_n \in \Gamma_1 \text{ with } |\check{P}(x_\gamma \otimes x_{\alpha_2} \otimes \dots \otimes x_{\alpha_n})| \neq 0\}$$

is countable, accoding to Lemma 4 (see below).

Assuming this, to finish the proof of Theorem 3.1, consider the non-countable set of indices obtained by removing from the set Γ_1 defined in (3.2), every index appearing in the countable set $S : \Gamma_2 = \Gamma_1 \backslash S$. Since S is countable, $A \backslash \Gamma_2$ is a countable set. Besides, whenever $\gamma_k \in \Gamma_2$ for $1 \leq k \leq n$, we have that $\check{P}(x_{\gamma_1} \otimes \dots \otimes x_{\gamma_n})$. In particular if $\gamma \in \Gamma_2$ then $x_\gamma \in \ker P$. Let us finally check that not only these elements, but the subspace generated by them, F_{Γ_2}, satisfies $F_{\Gamma_2} \subset \ker P$. Let $\gamma_1, \dots, \gamma_k \in \Gamma_2$ and $\lambda_1, \dots \lambda_k \in \mathbb{C}$. We obtain the result for the elements in the linear span of $\{x_\gamma, \gamma \in \Gamma_2\}$ from the following computation:

$$P(\lambda_1 x_{\gamma_1} + \dots + \lambda_k x_{\gamma_k}) = \sum_{\substack{i_1 = 1, \dots, k \\ \dots \\ i_n = 1, \dots, k}} \lambda_{i_1} \dots \lambda_{i_n} \check{P}(x_{\gamma_{i_1}} \otimes \dots \otimes x_{\gamma_{i_n}}) = 0.$$

The result for the closure F_{Γ_2} of the linear span is obtained just from the continuity of P.

□

Lemma 3. *Let $k, n \in \mathbb{N}$, and let $\{e_i, i \in \mathbb{N}\}$ be the set of coordinate vectors in l_∞. For any indices $i_m^j \in \mathbb{N}$, $j = 1, \dots, k$, $\quad m = 2, \dots, n$, the set*

$$\{e_{i_j^1} \otimes e_{i_j^2} \otimes \dots \otimes e_{i_j^n};\ 1 \leq j \leq k\} \tag{3.3}$$

defines a basis isometrically equivalent to the canonical basis of l_∞^k in $l_\infty \hat{\otimes}_\pi \overset{n}{\dots} \hat{\otimes}_\pi l_\infty$.

Lemma 4. *Consider Γ_1 the subset of A defined in* (3.2)*. Then, the set*

$$S = \{\Gamma_1;\ \textit{there are } \alpha_2, \dots, \alpha_n \in \Gamma_1 \textit{ with } \check{P}(x_\gamma \otimes x_{\alpha_2} \otimes \dots \otimes x_{\alpha_n}) \neq 0\}$$

is at most countable.

Corollary 4. *If $F \subset l_\infty$ is separable and $c_0 \subset F$, then F is not the intersection of any denumerable family of sets of zeroes of scalar-valued polynomials.*

The following theorem is an important consequence of Theorem 3.1. It asserts the existence of non-separable subspaces in the set of zeroes of every polynomial P on the complex l_∞ space, such that $P(0) = 0$.

Theorem 3.2. *Let E be a complex Banach space containing l_∞ and $P \in \mathcal{P}(E)$ be such that $P(0) = 0$. Then, there exists a non-separable subspace $F \subset$* ker *P.*

For a fixed Banach space E, $\quad n \in \mathbb{N}$, and a polynomial $P \in \mathcal{P}(^nE)$ with $P(0) = 0$ we say that the subspace $F \subset E$ is *maximal among the subspaces contained* in ker P (or just maximal if the context is clear) if $F \subset \ker P$ and whenever $G \subset E$ is any subspace satisfying $F \subset G \subset \ker P$, then necessarily $F = G$.

The following proposition describes an arbitrary maximal subspace for a homogeneous polynomial P as the intersection of the sets of zeroes of a finite number of polynomials (generally non-scalar). This result is particularly interesting for our purposes if the maximal subspace is separable.

Proposition 1. *Let E be a Banach space (real or complex), $n \in \mathbb{N}$, $P \in \mathcal{P}(^nE)$ and $F \subset E$ a subspace such that $F \subset \ker P$. Then F is maximal among the subspaces contained in* $\ker P$ *if and only if*

$$F = \bigcap_{k=1}^{n} \ker Q_k, \tag{3.4}$$

where $Q_k \in \mathcal{P}(^kE, \mathcal{P}(^{n-k}F))$ is defined for every $x \in E$ and every $y \in F$ as

$$Q_k(x)(y) = \hat{P}(x, \overset{k}{\ldots}, x, y, \overset{n-k}{\ldots}, y),$$

where $1 \le k \le n$.

Proof. Assume first that F is a maximal subspace, and consider any $x \in F$. By hypothesis $Q_n(x) = P(x) = 0$. For $1 \le k \le n-1$ we have $Q_k(x) = 0$ if and only if for every $y \in F$, $Q_k(x)(y) = 0$. The condition $P|_F = 0$ is equivalent to $\check{P}|_{F\times \overset{n}{\ldots} \times F} = 0$. Thus for every $y \in F$, $Q_k(x)(y) = \check{P}(x, \overset{k}{\ldots}, x, y, \overset{n-k}{\ldots}, y) = 0$. This implies that $x \in \ker Q_k$ and the contention $\subseteq$ in (3.4) is proved.

Observe that this proof does not make use of the maximality of F. However, to show the reverse inclusion this assumption is essential: Consider $x \in \bigcap_{k=1}^{n} \ker Q_k$ any scalar and $y \in F$. Then

$$\begin{aligned} P(\lambda x + y) &= \lambda^n P(x) + P(y) + \sum_{k=1}^{n-1} \lambda^k \begin{pmatrix} n \\ k \end{pmatrix} \check{P}(x, \overset{k}{\ldots}, x, y, \overset{n-k}{\ldots}, y) \\ &= \lambda^n P(x) + P(y) + \sum_{k=1}^{n-1} \lambda^k \begin{pmatrix} n \\ k \end{pmatrix} Q_k(x)(y) = 0. \end{aligned} \tag{3.5}$$

Since $F \subset [x] + F \subset \ker P$ and F is maximal, necessarily $x \in F$.

Assume now that $F \subset \ker P$ can be expressed as in (3.3). Let us prove that F is maximal. Consider $x \in \ker P$ such that $P(\lambda x + y) = 0$ for every scalar λ and every $y \in F$. Equation (3.5) still holds and says that for every fixed $y \in F$ we have a polynomial on $\lambda \in \mathbb{K}$ identically zero. Thus, every coefficient is zero. In this way it is proved that $x \in \ker Q_k$ for $k = 1, \ldots, n$. From expression (3.3) we get that $x \in F$ and consequently that F is maximal. $\square$

Corollary 5. *If $F \subset l_\infty$ is separable and $c_0 \subset F$, then F is not maximal for any $P \in \mathcal{P}(^nl_\infty)$.*

Observe that the description of a maximal separable subspace just given leads also to a proof of Theorem 3.2: As argued before, every complex polynomial with $P(0) = 0$ must be zero on a copy of c_0. This copy of c_0 is contained in a maximal subspace $F \subset \ker P$ which, by Corollary 5, is necessarily non-separable.

3.2. Zero Subspaces of Polynomials on $l_1(\Gamma)$

All results of this subsection was proved in [6].

The examples that we construct are defined on spaces $l_1(\Gamma)$. Over this space, all polynomials can be explicitly described. For instance, the general form of a quadratic functional $P : l_1(\Gamma) \to \mathbb{C}$ is

$$P(x) = \sum_{\alpha,\gamma\in\Gamma} \lambda_{\alpha\beta} x_\alpha x_\beta, \quad x = (x_\gamma)_{\gamma\in\Gamma} \in l_1(\Gamma),$$

where $(\lambda_{\alpha\beta})_{\alpha,\beta\in\Gamma}$ is a bounded family of complex scalars. Indeed in all our examples the coefficients $\lambda_{\alpha\beta}$ are either 0 or 1, they functionals of the form

$$P(x) = \sum_{\{\alpha,\beta\in G\}} x_\alpha x_\beta,$$

where G a certain set of couples of elements of Γ. On the other hand, we state the following elementary basic fact about polynomials that we shall explicitly use at some point.

Proposition 2. *Let $P : X \to Y$ be a homogeneous polynomial of degree n and norm K. Then*

$$||P(x) - P(y)|| \leq nKM^{n-1}||x||||y||$$

for every $x, y \in X$.

Let $\boldsymbol{\Omega}$ be a set and $\mathcal{A}$ be an almost disjoint family of subsets of $\boldsymbol{\Omega}$ (that is, $|A \cap A'| < |\boldsymbol{\Omega}|$ whenever $A, A' \in \mathcal{A}$ are different), and let $\mathcal{B} = \boldsymbol{\Omega} \cup \mathcal{A}$. We consider the following quadratic functional $P : l_1(\mathcal{B}) \to \mathbb{C}$ given by

$$P(x) = \sum\{x_n x_A : n \in \boldsymbol{\Omega}, \quad A \in \mathcal{A}, n \in A\}.$$

Theorem 3.3. *The space $X = l_1(\boldsymbol{\Omega}) \subset l_1(\mathcal{B})$ is maximal zero subspace for the polynomial P.*

Proof. The only point which requires explanation is that X is indeed maximal. So assume by contradiction that there is a vector y out of X such that $Y = \text{span}(X \cup \{y\})$ is a zero subspace for P. Without loss of generality, we suppose that y is supported in $\mathcal{A}$. Pick $A \in \mathcal{A}$ such that $|y_A| = \max\{|y_B| : B \in \mathcal{A}\}$ and $\mathcal{F} \subset \mathcal{A}$ a finite subset of $\mathcal{A}$ such that $\sum_{B\in\mathcal{A}\setminus\mathcal{F}} |y_B| < \frac{1}{9}|y_A|$. Now, because $\mathcal{A}$ is an almost disjoint family of subsets of $\boldsymbol{\Omega}$, it is possible to find $n \in \boldsymbol{\Omega}$ such that $n \in A$ but $n \notin B$ whenever $B \in \mathcal{F} \setminus \{A\}$. Consider the element $y + 1_n \in Y$. We claim that $P(y + 1_n) \neq 0$ getting thus a contradiction

$$P(y + 1_n) = \sum_{n\in B} y_B = y_A + \sum_{n\in B, B\in\mathcal{A}\setminus\mathcal{F}} y_B.$$

The second term of the sum has modulus less than $\frac{1}{9}$ the modulus of the first term. So $P(y + 1_n) \neq 0$. □

We are interested in the case when $|\mathcal{A}| > |\boldsymbol{\Omega}|$. The subspace $l_1(\mathcal{A})$ is a zero subspace for P. It may not be maximal but this does not matter because, by a Zorn's lemma argument, it is contained in some maximal zero subspace. This fact together with Theorem 3.3 shows that P has maximal zero subspaces of both densities $|\boldsymbol{\Omega}|$ and $|\mathcal{A}|$.

There are two standard constructions of big almost disjoin families. One is by induction, and it shows that for every cardinal κ we can find an almost disjoint family of cardinality κ^+ on a set of cardinality κ. The other one is by considering the branches of the tree $\kappa^{<\omega}$, and this indicates that for every cardinal κ we can find an almost disjoint family of cardinality κ^ω(one construction or the other provides a better result depending on whether $\kappa^\omega = \kappa$, $\kappa^\omega = \kappa^+$ or $\kappa^\omega > \kappa k^+$. Hence,

Corollary 6. *Let κ be an infinite cardinal and $\tau = \max(\kappa^+, \kappa^\omega)$. There exists a quadratic functional on $l_1(\tau)$ with a maximal zero subspace of density κ and another maximal zero subspace of density τ.*

Corollary 7. *There exists a quadratic functional on $l_1(\mathfrak{c})$ with a separable maximal zero subspace and a maximal zero subspace of density $\mathfrak{c}$.*

We denote by $[A]^2$ the set of all unordered pairs of elements of A

$$[A]^2 = \{t \subset A : |t| = 2\}.$$

We consider an ordinal α to be equal to the set of all ordinals less than α,so

$$\omega_1 = \{\alpha : \alpha < \omega_1\}$$

is the set of countable ordinals, and also for a nonnegative integer $n \in \mathbb{N}$, $n = \{0, 1, \dots, n-1\}$.

We introduce some notations for subsets of a well ordered set Γ. If $a \subset \Gamma$ is a set of cardinality n, and $k < n$ we denote by $a(k)$ the $(k+1)$-th element of a according to the well order of Γ, so that

$$a = \{a(0), \dots, a(n-1)\}.$$

Moreover, for $a, b \subset \Gamma$, we write $a < b$ if $\alpha < \beta$ for every $\alpha \in a$ and every $\beta \in b$.

We recall also that a Δ-system with root a is a family of sets such that the intersection of every two different elements of the family equals a. The well-known Δ-system lemma asserts that every uncountable family of finite sets has an uncountable subfamily which forms a Δ-system.

Definition 2. A function $f : [\Gamma]^2 \to 2$ is said to be a *partition of the first kind* if for every uncountable family A of disjoint subsets of Γ of some fixed finite cardinality n, and for every $k \in n$ there exist $a, b, a', b' \in A$ such that $f(a(k), b(k)) = 1$, $f(a'(k), b'(k)) = 0$ and $f(a(i), b(j)) = f(a'(i), b'(j))$ whenever $(i, j) \neq (k, k)$. Notice that, passing to a further uncountable subfamily A, we can choose such $a < b$ such that, in addition, $f(a(i), a(j)) = f(a'(i), a'(j)) = f(b(i), b(j)) = f(b'(i), b'(j))$ for all $\{i, j\} \in [n]^2$.

Theorem 3.4. *For $\Gamma = \omega_1$ there is a partition $f : [\Gamma]^2 \to 2$ of the first kind.*

Theorem 3.5. *If $f : [\Gamma]^2 \to 2$ is a partition of the first king and ıf Y is a subspace of $l_1(\Gamma)$ with $Y \subset \ker P_f$, then Y is separable.*

We shall denote by $\Delta_n = \{(i,i) : i \in n\}$ the diagonal of the cartesian product $n \times n$, $n = \{0, 1, \dots, n-1\}$. Also $B_X(x, r)$ or simply $B(x, r)$ will denote the ball of center x and radius r in a given Banach space X.

Definition 3. A function $f : [\Gamma]^2 \to \omega$ is said to be a *partition of the second kind* if for every uncountable family A of finite subsets of Γ all of some fixed cardinality n, we have the following two conclusions:

(a) there is an uncountable subfamily B of A and a function $h : n^2 \setminus \Delta_n \to \omega$ such that $f(a(i), b(j)) = h(i, j)$ for every $i \neq j$, $i, j < n$ and every $a < b$ in B;

(b) for every function $h : n \to \omega$ there exists $a < b$ in A such that $f(a(i), b(i)) = h(i)$.

Theorem 3.6. *For $\Gamma = \omega_1$ there is a partition $f : [\Gamma]^2 \to 2$ of the second kind as well.*

Theorem 3.7. *Suppose that $f : [\omega_1]^2 \to \omega$ is a partition of the second kind and let $P = P_f : l_1(\Gamma) \to V$ be the corresponding polynomial. Let Y be a nonseparable subspace of $l_1(\omega_1)$. Then $\overline{P(Y)}$ has nonempty interior in V.*

3.3. On the Zero-Set of Real Polynomials in Nonseparable Banach Spaces

All results of this subsection was proved in [12].

By $\mathcal{P}_f({}^nX)$ we denote the subspace of $\mathcal{P}({}^nX)$ formed by those polynomials which can be written as $P(x) = \sum\limits_{j=1}^{m} \lambda_j \langle u_j^*, x\rangle^n$, with $\lambda_j \in \mathbb{R}, u_j^* \in X^*, 1 \le j \le m$, and they are called *finite type* polynomials. The space of approximable polynomials, $\mathcal{P}_A({}^nX)$, is given by the closure of $\mathcal{P}_f({}^nX)$ in $\mathcal{P}({}^nX)$. By $\mathcal{P}_\omega({}^nX)$ we represent the subspace of $\mathcal{P}({}^nX)$ formed by those polynomials that are weakly continuous on the bounded subsets of X. A polynomial $P \in \mathcal{P}({}^nX)$ is a *nuclear* polynomial whenever it has the form $P(x) = \sum\limits_{j=1}^{\infty} a_j \langle u_j^*, x\rangle^n, \quad x \in X$, where $(a_j)_{j=1}^\infty$ and $(u_j^*)_{j=1}^\infty$ is a bounded sequence of X^*. Denoting by $\mathcal{P}_N({}^nX)$ the class of nuclear polynomials, it is quite clear that

$$\mathcal{P}_f({}^nX) \subset \mathcal{P}_N({}^nX) \subset \mathcal{P}_A({}^nX) \subset \mathcal{P}_\omega({}^nX) \subset \mathcal{P}({}^nX).$$

In what follows X will be an infinite-dimensional real Banach space and X^* its topological dual. We use the symbol $\langle\cdot,\cdot\rangle$ to denote the standard duality between X and X^*.

If $A \subset X$ and $B \subset X^*$, then we use the notation

$$A^\perp = \{x \in X : \langle x^*, \quad x\rangle = 0, x \in A\}, \quad B_\perp = \{x \in X : \langle x^*, x\rangle = 0, \quad x^* \in B\}.$$

For a polynomial $P \in \mathcal{P}({}^nX)$, the following conjugacy relationship between its first and $(n-1)$-th derivatives turned out to be relevant. The first derivative is the mapping $P' : X \to X^*$ such that

$$P'(x) = n\check{P}(x, \overset{(n-1)}{\ldots}, x, \cdot),\ x \in X,$$

while the $(n-1)$-th derivative is given by the continuous linear map $P^{(n-1)} : X \to \mathcal{L}_s(X^{n-1})$ such that

$$P^{(n-1)}(x) = n!\check{P}(x, \cdot, \overset{(n-1)}{\ldots}, \cdot), \quad x \in X,$$

where $\mathcal{L}_s(X^{n-1})$ denotes the space of symmetric continuous $(n-1)$-linear functionals on X. It is then straightforward to notice, using the Polarization formula, that

$$\ker P^{(n-1)} = P'(X)_\perp.$$

If Z is such a maximal subspace, then, for $x \in \ker P^{(n-1)}, \quad z \in Z$,

$$\begin{aligned} P(x+z) &= P(x) + P(z) + \sum_{j=1}^{n-1} \binom{n}{j} \check{P}(x, \overset{(j)}{\ldots}, x, z, \overset{(n-j)}{\ldots}, z) \\ &= \frac{1}{n!} P^{(n-1)}(x)(x, \overset{(n-1)}{\ldots}, x) + \sum_{j=1}^{n-1} \frac{1}{j!(n-j)!} P^{(n-1)}(x)(x, \overset{(j-1)}{\ldots}, x, z, \overset{(n-j)}{\ldots}, z) = 0, \end{aligned}$$

i.e., $Z + \ker P^{(n-1)} \subset \ker P$, and the maximality of Z yields that $\ker P^{(n-1)}$ is contained in Z. Hence, if $\ker P^{(n-1)}$ were non-zero, we would easily obtain a non-zero linear subspace contained in $P^{-1}(0)$. Indeed, we will seek for conditions in order to guarantee that $\ker P^{(n-1)}$ is sufficiently big. For this purpose, recall that

$$(\ker P^{(n-1)})^{\perp} = (P'(X)_{\perp})^{\perp} = \overline{lin}^{\omega^*}(P'(X)),$$

and so, roughly speaking, the smaller $P'(X)$ is the bigger $\ker P^{(n-1)}$ will be. In particular, if $P'(X)$ were separable, then $(X/\ker P^{(n-1)})^* = (\ker P^{(n-1)})^{\perp}$ would have to be weak*-separable and this is mainly the reason why in the next section we shall be dealing with this type of space.

We say that a real Banach space X is in class $\mathcal{C}_H$ whenever there exists a one-to-one continuous linear map from X into a Hilbert space. When $X \in \mathcal{C}_H$ we shall say that X is injected into a Hilbert space. If X is injected into a separable Hilbert space, then we shall write $X \in \mathcal{W}^*$. Clearly, $\mathcal{W} \subset \mathcal{C}_H$. The following properties of the spaces in these two classes are quite straightforward.

Proposition 3. *The following conditions are equivalent for a space X :*

(i) $X \in \mathcal{W}^*$.

(ii) X^* is weak*-separable.

(iii) X^* has a countable total subset.

Proposition 4. *If X is in class $\mathcal{C}_H$ (respectively, in $\mathcal{W}^*$) and Y is a space that is injected linearly and continuously into X, then $Y \in \mathcal{C}_H$ (respectively, $Y \in \mathcal{W}^*$). Hence, every closed linear subspace of X is in the same class that X.*

Proposition 5. *If X is separable, then X and X^* are in $\mathcal{W}^*$.*

Proposition 6. *Let Y be a closed linear subspace of the Banach space X. If Y is in $\mathcal{W}^*$ and X/Y is in $\mathcal{C}_H$, then X is in $\mathcal{C}_H$.*

Proof. With no loss of generality, we may assume that we have two one- to-one bounded linear maps

$$S_1 : Y \to l_2, \quad S_2 : X/Y \to l_(\Gamma_0),$$

with Γ_0 being a set that is disjoint from the set of positive integers $\mathbb{N}$. Now, for each $j \in \mathbb{N}$, if e_j denotes the corresponding unit vector, we have that $S_1^* e_j \in Y^*$. Let $v_j^* \in X^*$ be the extension of $S_1^* e_j$ to X such that $||v_j^*|| = ||S_1^* e_j||$. Setting $\Gamma := \mathbb{N} \cup \Gamma_0$, we define the mapping $T : X \to l_2(\Gamma)$ such that, for $x \in X$, $\quad Tx := (\lambda_\gamma)_{\gamma \in \Gamma}$ where

$$\lambda_\gamma := \begin{cases} 2^{-\gamma}\langle v_\gamma^*, x\rangle, & \gamma \in \mathbb{N}, \\ \langle S_2(x+Y), e_\gamma\rangle, & \gamma \in \Gamma_0. \end{cases}$$

Then, T is a well defined linear map such that it is bounded. To see that it is one-to-one, let $x \in X$ be such that $Tx = 0$, then, $0 = \langle S_2(x+Y), e_\gamma\rangle, \quad \gamma \in \Gamma_0$, implies that $S_2(x+Y) = 0$, and so $x \in Y$; hence, from $0 = 2^{-j}\langle v_j^*, x\rangle, \quad j \in \mathbb{N}$, it follows that $0 = \langle S_1^* e_j, x\rangle = \langle e_j, S_1 x\rangle, \quad x \in \mathbb{N}$, therefore $S_1 x = 0$, and $x = 0$. ☐

Corollary 8. *Let Y be a closed linear subspace of X such that Y and X/Y are both in $\mathcal{W}^*$, then X is also in $\mathcal{W}^*$, i.e., being in $\mathcal{W}^*$ is a three-space property.*

We already know that, if X is separable then X and X^* are both in $\mathcal{W}^*$, let's take a look now at some other examples of spaces not belonging to $\mathcal{W}^*$, which will obviously be non-separable. Every non-separable weakly compactly generated space, and hence every non-separable reflexive one and $c_0(\Gamma)$, Γ an uncountable set, has a non-weak*-separable dual. This, plus the fact that $c_0(\Gamma)$ can be canonically injected into $l_\infty(\Gamma)$, yields that, for uncountable Γ, $c_0(\Gamma)$ and $l_\infty(\Gamma)$ are not in $\mathcal{W}^*$ and clearly $l_2(\Gamma) \in \mathcal{C}_H\mathcal{W}^*$. The easiest example of a space X such that $X \in \mathcal{W}^*$ and $X^* \notin \mathcal{W}^*$ is given by $X = l_\infty$: Being obvious that $l_\infty \in \mathcal{W}^*$, we show that $l_\infty^* \notin \mathcal{W}^*$: l_∞ contains a closed subspace F such that l_∞/F is isomorphic to a non-separable Hilbert space. Hence, $F^\perp = (l_\infty/F)^*$ is a subspace of l_∞^* which is also isomorphic to a non-separable Hilbert space. If l_∞^* were in class $\mathcal{W}^*$, then, from Proposition 4, there would be a non-separable Hilbert space in $\mathcal{W}^*$, which is clearly contradictory.

There are also examples satisfying the contrary, i.e., $X \notin \mathcal{W}^*$ and $X^* \in \mathcal{W}^*$. In particular, there is one which plays a somewhat outstanding role and we shall take a look at it right now. Let $X = c_0([0,1])$. Then $X^* = l_1([0,1])$, and to show that X^* is in $\mathcal{W}^*$, since the space of continuous functions $C[0,1]$, being separable, is a quotient of l_1, and therefore its topological dual $C[0,1]^*$ is isomorphic to a subspace of l_∞, it suffices to see that $l_1([0,1])$ can be continuously injected into $C[0,1]^*$. This is done by noticing that the mapping $T : l_1([0,1]) \to C[0,1]^*$ such that, if $x = (x_\gamma) \in l_1([0,1])$, $\quad Tx := \sum_{\gamma\in[0,1]} x_\gamma\delta_\gamma$ where δ_γ is the Dirac measure at the point $\gamma \in [0,1]$, is one-to-one bounded and linear.

Also, since $(l_\infty/c_0)^*$ admits no countable total subsets, it follows that l_∞/c_0 is not in $\mathcal{W}^*$.

Let us to show that, if $X \notin \mathcal{W}^*$, then every sequence of closed linear subspaces $(E_j)_{j=1}^\infty$ such that $X/E_j \in \mathcal{W}^*$, $j \geq 1$, satisfies that $\bigcap_{j=1}^\infty E_j \notin \mathcal{W}^*$.

Lemma 5. *Let E be a closed linear subspace of the Banach space X. Then $E^\perp$ is $\sigma(X^*,X)$-separable if and only if there is a sequence $(u_j^*)_{j=1}^\infty$ in X^* such that $E = \cap_{j=1}^\infty \ker u_j^*$.*

Proposition 7. *Let $(E_j)_{j=1}^\infty$ be a sequence of closed linear subspaces of X such that, for each j, $E_j^\perp$ is $\sigma(X^*,X)$-separable. Let $E := \cap_{j=1}^\infty E_j$, then:*

(i) $E^\perp$ is also $\sigma(X^*,X)$-separable.

(ii) If $X\sigma \notin \mathcal{W}^*$, then $E \notin \mathcal{W}^*$.

Proof. For each j, from the previous lemma, there is a sequence $(u_{jk}^*) \subset X^*$ such that $E_j = \bigcap_{k=1}^\infty \ker u_{jk}^*$. Hence $E^\perp = (\cap_{j=1}^\infty E_j)^\perp = (\cap_{j,k=1}^\infty \ker u_{jk}^*)^\perp = \overline{lin}^{w^*}\{u_{jk}^* : j,k \geq 1\}$ is clearly $\sigma(X,X^*)$-separable, thus obtaining (i). Besides, this yields $X/E \in \mathcal{W}^*$, and, if $X \notin \mathcal{W}^*$ the 3-space property shown in Corollary 8 guarantees (ii). ☐

Proposition 8. *If X is a Banach space which is not in class $\mathcal{W}^*$, then, if n is any positive integer, for each $P \in \mathcal{P}_w(^nX)$, $\ker P^{(n-1)}$ is not in $\mathcal{W}^*$.*

Proof. If $P \in \mathcal{P}_w(^nX)$, making use of the conjugacy relation mentioned in the first section, we have

$$\ker P^{(n-1)} = P'(X)_\perp.$$

From ([10], p. 88, Proposition 2.6), we know that P' is (weak-to-norm)-uniformly continuous on the bounded subsets and, since B_X is weakly precompact, it follows that $P'(X)$ is norm-separable in X^*. Clearly then, $\overline{\text{lin}}^{w^*}(P'(X))$ is weak*-separable and so, since

$$(X/P'(X)_\perp)^* = (P'(X)_\perp)^\perp = \overline{\text{lin}}^{w^*}(P'(X)),$$

we have that $X/P'(X)_\perp$ is in $\mathcal{W}^*$. From Corollary 8, since X is not in $\mathcal{W}^*$, it follows that $\ker P^{(n-1)} = P'(X)_\perp$ is not in $\mathcal{W}^*$. □

Recalling that $\ker P^{(n-1)}$ is contained in every maximal linear subspace contained in $\ker P$, the next result clearly follows.

Corollary 9. *If $X \notin \mathcal{W}^*$, then, for every integer n and every $P \in \mathcal{P}_w(^nX)$, every maximal linear subspace Z contained in $\ker P$ is such that $Z \notin \mathcal{W}^*$.*

The next result gives us another characterization of the spaces in class $\mathcal{W}^*$.

Corollary 10. *For a Banach space X, the following conditions are equivalent:*

(i) $X \in \mathcal{W}^*$.

(ii) For any even integer n, X admits a positive definite polynomial $P \in \mathcal{P}_N(^nX)$.

(iii) For any even integer n, X admits a positive definite polynomial $P \in \mathcal{P}_w(^nX)$.

(iv) There is an even integer n such that X admits a positive definite polynomial $P \in \mathcal{P}_w(^nX)$.

(v) There is an even integer n such that X admits a positive definite polynomial $P \in \mathcal{P}_N(^nX)$.

As a by product of this last corollary, the author obtained a stronger version of part (i) in Theorem 16 of [1].

Corollary 11. *Let X be any infinite-dimensional real Banach space. Then, either X admits a positive definite nuclear polynomial of degree 2, or, for every positive integer n, the zero-set of every $P \in \mathcal{P}_w(^nX)$ contains a closed linear subspace of X whose dual is not weak*-separable.*

The results previously obtained will be used in the following to show that, if $X \notin \mathcal{W}^*$, then every vector-valued polynomial, not necessarily homogeneous, which is weakly continuous on the bounded subsets of X admits a closed linear subspace not belonging to $\mathcal{W}^*$ where the polynomial is constant.

Lemma 6. *If $P \in \mathcal{P}_w(^nX)$, then $(\ker P^{(n-1)})^\perp$ is $\sigma(X^*, X)$-separable.*

Proposition 9. *Let $(n_j)_{j=1}^\infty$ be a sequence of positive integers and let $(P_j)_{j=1}^\infty$ be a sequence of polynomials such that, for each j, $P_j \in \mathcal{P}_w(^{n_j}X)$. If $X \in \mathcal{W}^*$, then there is a closed linear subspace Z in X such that $Z \notin \mathcal{W}^*$ and $Z \subset \cap_{j=1}^\infty P_j^{-1}(0)$.*

Proof. For each j, set $Z_j := \ker P_j^{(n_j-1)}$. Then, from the previous lemma we have that $Z_j^\perp$ is $\sigma(X^*, X)$-separable, $j \geq 1$. Setting $Z := \cap_{j=1}^\infty Z_j$, we know from Proposition 7 that $Z^\perp$ is (X^*, X)-separable, and so, since $X \notin \mathcal{W}^*$, we have that $Z \notin \mathcal{W}^*$. Now, since it is evident that $\ker P_j^{(n_j-1)} \subset \ker P_j$, $j \geq 1$, the result follows. □

For a Banach space Y and a positive integer n, the symbols $\mathcal{P}(^nX,Y)$ and $\mathcal{P}_w(^nX,Y)$ will denote the spaces of n-homogeneous continuous polynomials on X with values in Y and the subspace formed by those which are weakly continuous (to say it in a more explicit way, weak-to-norm continuous) on the bounded subsets of X, respectively. We see next that, when X is not in class $\mathcal{W}^*$, any countable family of polynomials in $\mathcal{P}_w(^nX,Y)$ vanishes simultaneously on quite a big linear subspace.

Corollary 12. *Let X, Y be Banach spaces with $X \notin \mathcal{W}^*$. Let $(n_j)_{j=1}^{\infty}$ be a sequence of positive integers and $(P_j)_{j=1}^{\infty}$ a sequence of polynomials such that, for each j, $P_j \in \mathcal{P}_w(^{n_j}X,Y)$. Then there is a closed linear subspace Z in X such that $Z \notin \mathcal{W}^*$ and $P_j|Z = 0, \quad j \geq 1$.*

Corollary 13. *Let $P : X \to Y$ be a polynomial, not necessarily homogeneous, which is weakly continuous on the bounded subsets of X. If $X \notin \mathcal{W}^*$, then there is a closed linear subspace Z in X such that $Z \notin \mathcal{W}^*$ and $P|_Z = P(0)$.*

In the results previously given we determine constructively the big linear subspace contained in the polynomial's zero-set. Nevertheless, noticing that what we really use is that weak zero-neighborhoods contain finite-codimensional linear subspaces, there is a natural extension of these existence results to a larger frame, namely that of the mappings which are weak-to-norm continuous on the bounded sets. More explicitelly, we have the following generalization.

Corollary 14. *Let $f : X \to Y$ be a weak-to-norm continuous mapping on the bounded subsets of X such that $f(0) = 0$. If $X \notin \mathcal{W}^*$, then there is a closed linear subspace Z in X, with $Z \notin \mathcal{W}^*$, such that $Z \subset f^{-1}(0)$.*

Corollary 15. *Let $(f_j)_{j=1}^{\infty}$ be a sequence of mappings from X into Y which are weak-to-norm continuous on the bounded subsets of X and such that $f_j(0) = 0, \quad j \geq 1$. Assume that, for all $x \in X$,*

$$f(x) := \lim_j f_j(x)$$

exists.

Conjecture. For a real Banach space X, either $X \in \mathcal{C}_H$, or, for every $P \in \mathcal{P}(^2X)$, $\ker P$ contains a non-separable linear subspace.

Proposition 10. *Let X be a space such that $X \notin \mathcal{C}_H$ and $X^* \in \mathcal{C}_H$. Then, if $P \in \mathcal{P}(^2X)$,* $\ker P' \notin \mathcal{W}^*$.

Proof. The first Fréchet derivative of P is the continuous linear map $P' : X \to X^*$ such that $\langle P'(x), y\rangle = 2\check{P}(x,y), \ x, y \in X$. Assuming $\ker P'$ were in $\mathcal{W}^*$, then, from Proposition 6, since $X \notin \mathcal{C}_H$, we would have that $X/\ker P' \notin \mathcal{C}_H$. But the map $T : X/\ker P' \to X^*$ given by $T(x + \ker P') := P'(x)$ is well defined linear bounded and one-to-one, which would imply that $X/\ker P'$ is injected into X^*, but $X^* \in \mathcal{C}_H$, after Proposition 4, would then yield $X/\ker P' \in \mathcal{C}_H$, a contradiction. □

Corollary 16. *If $X \notin \mathcal{C}_H$ and $X^* \in \mathcal{C}_H$, then, for every $P \in \mathcal{P}(^2X)$, every maximal linear subspace Z contained in* $\ker P$ *is such that $Z \notin \mathcal{W}^*$.*

We show next that, for uncountable Γ, the spaces $c_0(\Gamma)$, $l_p(\Gamma)$, $\quad 2 < p < \infty$, are of the type just considered, i.e., $X \notin \mathcal{C}_H, \quad X^* \in \mathcal{C}_H$.

Lemma 7. *Let Γ be an uncountable set. Then, for $1 \leq p \leq 2$, the space $l_p(\Gamma) \in \mathcal{C}_H$, while, for $2 < p \leq \infty, \quad l_p(\Gamma) \notin \mathcal{C}_H$.*

Corollary 17. *Let Γ be an uncountable set and let X be any of the spaces $l_p(\Gamma)$, $2 < p < \infty$, or $c_0(\Gamma)$. If P is a continuous 2-homogeneous polynomial on X, then $\ker P'$ is a closed linear subspace contained in $\ker P$ whose dual is not weak*-separable. Consequently, every maximal linear subspace contained in $\ker P$ has a dual which is not weak*-separable. If $X = l_\infty(\Gamma)$, then, for every $P \in \mathcal{P}(^2X)$, $\ker P$ contains a closed linear subspace Z such that $Z \notin \mathcal{W}^*$.*

We say that a space X is in class $\mathcal{C}'_H$ whenever, for any sequence $(u_j^*)_{j=1}^\infty$ in X^*, we have that $\bigcap_{j=1}^\infty \ker u_j^* \notin \mathcal{C}_H$.

Clearly, $\mathcal{C}_H$ and $\mathcal{C}'_H$ are disjoint classes and we show that for the elements of class $\mathcal{C}'_H$ the conjecture holds.

Proposition 11. *Let $X \in \mathcal{C}'_H$. If $P \in \mathcal{P}(^2X)$, then every maximal linear subspace contained in $\ker P$ is non-separable.*

Proof. Let Z be one of such maximal subspaces and suppose it is separable. Let $Y := P'(Z)_\perp$. Then, by the maximality of Z, we have that $\ker P \cap Y = Z$ and P does not change sign in Y (we shall assume that $P|_Y \geq 0$).

Since $Y^\perp = \overline{P'(Z)}^{w^*}$ is $\sigma(X^*, X)$-separable, after Lemma 5 we have that there is a sequence $(u_j^*)_{j=1}^\infty$ in X^* such that $Y = \cap_{j=1}^\infty \ker u_j^*$. Thus, since $X \in \mathcal{C}'_H$, it follows that $Y \notin \mathcal{C}_H$. Now, by defining

$$Q(x + Z) := P(x),\ x \in Y,$$

we obtain a polynomial $Q \in \mathcal{P}(^2(Y/Z))$ which is positive definite. This implies that $Y/Z \in \mathcal{C}_H$, but, Z being separable yields $Z \in \mathcal{W}^*$, and so, after Proposition 6, we have that $Y \in \mathcal{C}_H$, a contradiction. $\square$

4. The Real Case

All results of this subsection was proved in [1].

Let E be a real Banach space. The author showed that either E admits a positive definite 2-homogeneous polynomial or every 2-homogeneous polynomial on E has an infinite dimensional subspace on which it is identically zero. Under addition assumptions, he showed that such subspaces are non-separable. He examined analogous results for nuclear and absolutely (1,2)-summing 2-homogeneous polynomials and give necessary and sufficient conditions on a compact set K so that $C(K)$ admits a positive definite 2-homogeneous polynomial or a positive definite nuclear 2-homogeneous polynomial.

The case of the polynomial $P : \mathbb{R}^n \to \mathbb{R}$, $\quad P(x) = \sum\limits_{j=1}^{n} x_j^2$ not with standing, it is exactly the zeros of real valued 2-homogeneous polynomials which will be of interest here, in the case when the domain $\mathbb{R}^n$ is replaced by an infinite dimensional real Banach space E. There are many "large" Banach spaces E for which there is no positive definite 2-homogeneous polynomial P. As we will see, for a real Banach space E, either E admits a positive definite 2-homogeneous polynomial or every 2-homogeneous polynomial on E is identically zero on an infinite dimensional subspace of E.

We recall that an n-homogeneous polynomial $P : E \to \mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is, by definition, the restriction to the diagonal of a necessarily unique symmetric continuous n-linear form $\check{P} : E \times$

$\ldots \times E \to \mathbb{K}$; that is, $P(x) = \check{P}(x, ..., x)$ for every $x \in E$. The polynomial P is said to be *positive definite* if $P(x) \geqslant 0$ for every x and $P(x) = 0$ implies that $x = 0$.

An n-homogeneous polynomial P on E is *nuclear* if there is bounded sequence $(\phi_j)_{j=1}^{\infty} \subset E'$ and a point $(\lambda_j)_{j=1}^{\infty}$ in l_1 such that

$$P(x) = \sum_{j=1}^{\infty} \lambda_j \phi_j(x)^n$$

for every x in E. The space of all nuclear n-homogeneous polynomials on E is denoted by $\mathcal{P}_N({}^nE)$. A sequence $(x_j)_j$ in E is said to be *weakly* $2-$ *summing* if

$$\sup_{\phi \in B_{E'}} \sum_{j=1}^{\infty} \phi(x_j)^2 < \infty.$$

An n-homogeneous polynomial P on E is said to be (absolutely) (1,2)-summing if P maps weakly 2-summing sequences into absolutely summable sequences; that is if $\sum\limits_{j=1}^{\infty} ||P(x_j)|| < \infty$ for every weakly 2-summing sequence $(x_j)_j$. P is (1,2)-summing if and only if there is $C > 0$ so that for every positive integer m and every $x_1, \ldots, x_m$ in E we have

$$\sum_{j=1}^{m} |P(x_j)| \leqslant C \left(\sup_{\phi \in B_{E'}} \sum_{j=1}^{m} \phi(x_j)^2 \right)^{\frac{n}{2}}.$$

Proposition 12. *A polynomial $P \in \mathcal{P}({}^2E)$ is positive definite if and only if for every $x, y \in E$ such that $x \neq \pm y$,*

$$|\check{P}(x, y)| < \frac{1}{2}(P(x) + P(y)).$$

Consequently, if P is a positive definite 2-homogeneous polynomial on E, then $||P|| = ||\check{P}||$.

Proof. Assume that P is positive definite, and so $\check{P}$ is an inner product. Hence we may apply the Cauchy-Schwarz inequality: $|\check{P}(x, y)| \leq |P(x)P(y)|^{\frac{1}{2}}$, with equality if and only if $x = \pm y$. Next, by the arithmetic-geometric inequality, $|P(x)P(y)|^{\frac{1}{2}} \leq \frac{1}{2}(P(x) + P(y))$. The converse follows by taking an arbitrary $x \neq 0$ and $y = 0$ in the inequality. □

Proposition 13. *The following conditions on a Banach space E are equivalent:*

(i) E admits a positive 2-homogeneous polynomial.

(ii) There is a continuous linear injection from E into a Hilbert space.

(iii) The point 0 is an exposed point of the convex cone of the subset $\{\delta_x \bigotimes \delta_x : x \in S_E\}$ of the symmetric tensor product $E \bigotimes_{\pi,s} E$, where S_E is the unit sphere of E.

(iv) There is a 2-homogeneous polynomial P on E whose set of zeros is contained in a finite dimensional subspace of E.

Proof. (i) $\Rightarrow$ (ii): Let $\check{P}$ be the symmetric positive definite bilinear form associated to the positive definite polynomial P, so that $(E, \check{P})$ is a pre-Hilbert space with completion, say, H with the induced pre-Hilbert norm. Then the injectionj : $E \to H$ is continuous since $||j(x)|| = |\check{P}(x,x)|^{\frac{1}{2}} = |P(x)|^{\frac{1}{2}} \le ||P||^{\frac{1}{2}}||x||$.

(ii) $\Rightarrow$ (iii): Note that the space of 2-homogeneous polynomials on E is the dual of $E\widehat{\bigotimes}_{\pi,s}E$. Also, recall that the convex cone of the set $\{\delta_x \bigotimes \delta_x : x \in S_E$ consists of all points of the form $\{\sum_{i=1}^n a_i\delta_{x_i} \bigotimes \delta_{x_i},$ where $x_i \in S_E$ and $a_i \ge 0\}$. Now, the polynomial $P(x) \equiv \langle j(x), j(x)\rangle$ is positive definite on E. If we regard P as an element of $(E\widehat{\bigotimes}_{\pi,s}E)'$, we see at $P(0) = 0$ while $P(\delta_x \bigotimes \bigotimes \delta_x) = P(x) > 0$ for all $x \in S_E$. Consequently, for any point $\sum\limits_{i=1}^n a_i\delta_{x_i} \bigotimes \delta_{x_i}$ in the convex cone, $P(\sum\limits_{i=1}^n a_i\delta_{x_i} \bigotimes \delta_{x_i}) = \sum\limits_{i=1}^n a_iP(x_i) \ge 0$, with equality if and only if all $a_i = 0$.

(iii) $\Rightarrow$ (iv): Let $T \in (E\widehat{\bigotimes}_{\pi,s}E)'$ be such that $T(0) = 0$ and $T(b) > 0$ for all b in the convex cone. In particular, for all $x \in S_E$, $P(x) \equiv T(\delta_x \bigotimes \delta_x) > 0$, so that $\ker P = 0$.

(iv) $\Rightarrow$ (i): We only consider the non-trivial situation, when $\dim E = \infty$. Suppose that P is a 2-homogeneous polynomial whose zero set is contained in the finite dimensional subspace V with basis, say, $\{v_1, \dots, v_n\}$. We first observe that $P(x)$ is always positive or always negative, for all $x \in E\backslash V$. Otherwise, there would exist $x, y \in S_{E\backslash V}$ such that $P(x) < 0 < P(y)$. Let $\gamma : [0,1] \to E\backslash V$ be a curve linking x and y. Then $P \circ \gamma(t) = 0$ for some $t \in [0,1]$, which is a contradiction. So, without loss of generality, we assume that $P(x) \ge 0$ for all $x \in E$. Let $\Pi : E \to V$ be a projection, with $\Pi(x) = \sum_{i=1}^n a_i(\Pi(x))v_i$. Then, the 2-homogeneous polynomial Q defined by $Q(x) \equiv P(x) + \sum_{i=1}^n a_i(\Pi(x))^2$ is positive definite. □

Remark 1. Suppose that there is a normalized sequence $(\phi_j)_j \in E'$ such that if $x \in E, \phi_j(x) = 0$ for all j, then $x = 0$. Then the mapping $x \in E \mapsto (\frac{1}{j}\phi_j(x))$ defines an injection into l_2, and so Proposition 13 applies. In particular, any separable space and $C(K)$spaces, when K is compact and separable, admit a positive definite 2-homogeneous polynomial. On the other hand $E = c_0(\Gamma)$ and $E = l_p(\Gamma)$, where Γ is an uncountable index set and $p > 2$, do not admit positive definite 2-homogeneous polynomials.

We also note that if there is a continuous linear injection $j : E \to l_2$, $j(x) = (j_n(x))$, then the mapping $x \mapsto (\frac{j_n(x)}{2^n})$ is a *nuclear* injection between these spaces. We have proved $(ii) \Rightarrow (iii)$ of the following separable version of Proposition 13.

Proposition 14. *Let E be a real Banach space. The following conditions are equivalent:*

(i) E admits a positive definite 2-homogeneous nuclear polynomial.

(ii) E admits a continuous injection $j : E \to l_2$.

(iii) There is a nuclear injection $j : E \to l_2$ *of the form* $j(x) = \sum\limits_{n=1}^{\infty} \chi_n(x)e_n$ *with* $(||\chi_n||) \in l_1$.

Proof. (i) $\Rightarrow$ (ii): If $P(x) = \sum_{n=1}^{\infty} \phi_n(x)^2$ is a positive definite nuclear polynomial on E, then $j(x) = \sum_{n=1}^{\infty} \phi_n(x)e_n$ will satisfy (ii).

(iii) $\Rightarrow$ (i): Let $j : E \to l_2$ be a nuclear injection, $j(x) = \sum_{n=1}^{\infty} \chi_n(x)e_n$, where $(||\chi_n||)_n \in l_1$. Since $\bigcap_{n=1}^{\infty} \ker \chi_n = \{0\}$, it follows that the 2-homogeneous polynomial $P : E \to \mathbb{R}$, $P(x) = \sum_{n=1}^{\infty} \chi_n^2(x)$ is positive definite. Finally, P is nuclear since $\sum_{n=1}^{\infty} ||\chi_n||^2 \leq \sum_{n=1}^{\infty} ||\chi_n|| < \infty$. □

Theorem 4.1. *Let E be a real Banach space which does not admit a positive definite 2-homogeneous polynomial. Then, for every $P \in \mathcal{P}(^2E)$, there is an infinite dimensional subspace of E on which it is identically zero.*

Proof. Suppose E does not admit a positive definite 2-homogeneous polynomial and that $P \in \mathcal{P}(^2E)$. Let $\mathcal{S} = \{S : S \text{ is a subspace of } E \text{ and } P|_S \equiv 0\}$. Order $\mathcal{S}$ by inclusion and use Zorn's Lemma to deduce the existence of a maximal element S of $\mathcal{S}$. Suppose that S is finite dimensional. $v_1, \ldots, v_n$ be a basis for S and let $T = \bigcap_{x \in S} \ker A_x = \bigcap_{i=1}^{n} \ker A_{v_i}$ where $A_x : E \to \mathbb{R}$ is the linear map which sends y in E to $\check{P}(x, y)$. We note that $S \subset T$. To see this suppose that $y \in S$. Then for every $s \in S$, $s + y$ is also in S. Since

$$0 = P(s + y) = P(s) + 2A_s(y) + P(y) = 2A_s(y)$$

for every $s \in S$ we see that $y \in T$.

Since S is finite dimensional we can write T as $T = S \bigoplus Y$ for some subspace Y of T. It is easy to see that all the zeros of $P|_T$ are contained in S. Therefore, either $P|_T$ or $-P|_T$ is positive definite on Y. Let us suppose, without loss of generality, that $P|_T$ is positive definite on Y. As S is n-dimensional we can find $\phi_1, \ldots, \phi_n$ so that $P + \sum_{i=1}^{n} \phi_i^2$ is positive definite on T. Note that T has finite codimension in E and hence is complemented. Let π_T be the (continuous) projection of E onto T. Then $(P + \sum_{i=1}^{n} \phi_i^2) \circ \pi_T + \sum_{i=1}^{n} A_{v_i}^2$ is a positive definite polynomial on E, contradicting the fact that E does not admit such a polynomial. □

Theorem 4.2. *Let E be a real Banach space of type 2. Then either E admits a positive definite 2-homogeneous polynomial or every $P \in \mathcal{P}(^2E)$ has an non-separable subspace on which it is identically zero.*

Proof. Assume that E does not admit a positive definite 2-homogeneous polynomial and let $P \in \mathcal{P}(^2E)$. Let $S \subset E$ be a maximal subspace such that $P|_S \equiv 0$. If S is separable, the argument in Theorem 4.1 shows that the subspace $T \subset E$ can be written $T = S \bigoplus_a Y$, where Y is an algebraic complement of S in T and where, without loss of generality, $P|_T$ is positive definite on Y. Then for every $s \in S$ and $t \in T$:

$$P(s + t) = P(s) + 2\check{P}_s(t) + P(t) = P(t) \geq 0.$$

Since S is separable, we can find a sequence $\{\phi_i\}_{i=1}^{\infty}$ in E' so that $\sum_{i=1}^{\infty} \phi_i^2$ is positive definite on S, and hence $P + \sum_{i=1}^{\infty} \phi_i^2$ is positive definite on T. Hence we have a continuous linear injection i of T into some Hilbert space $L_2(I)$. Since E is type 2, Maurey's Extension Theorem ([8], Theorem 12.22) allows us to extend i to a (not necessarily injective) linear map $\widetilde{i}$ from E into $L_2(I)$. Finally, define a map j from E into $L_2(I) \bigoplus_{l_2} l_2$ by

$$j(x) = \left(\widetilde{i}(x), \sum_{i=1}^{\infty} \frac{A_{v_i}(x)}{i^2 ||A_{v_i}||} e_i \right),$$

where e_i is the i^{th} basis vector in l_2. Since j is a continuous injection, E admits a positive definite polynomial, which is a contradiction. □

Theorem 4.3. *Let E be a real Banach space which does not admit a positive definite 4-homogeneous polynomial. Then for every 2-homogeneous polynomial P on E, there is a non-separable subspace of E on which P is identically zero.*

Theorem 4.4. *Let E be a real Banach space which does not admit a positive definite 4-homogeneous polynomial, and let $(\psi_k)_{k=1}^{\infty}$ be a sequence in E'. Then for any countable family $(P_j)_{j=1}^{\infty} \subset \mathcal{P}(^2E)$, there is a non-separable subspace of $\bigcap_{k=1}^{\infty} \ker \psi_k$ on which each P_j is identically zero.*

Note that if E does not admit a positive definite 4-homogeneous polynomial, then it cannot admit a positive definite 2-homogeneous one either. An example of an E satisfying the hypotheses of Theorems 4.3 and 4.4 is $E = l_p(I)$, where I is an uncountable index set and $p > 4$.

Proof. **of Theorem 4.4**: The argument begins in a similar way to our earlier proofs. As before, let S be a maximal element of $\mathcal{S} = \{S : S \text{ is a subspace of } \bigcap_{k=1}^{\infty} \ker \psi_k \text{ and } P_j|_S \equiv 0, \text{all } j\}$. Suppose that S is separable, with countable dense set $(v_i)_{i=1}^{\infty}$. Let $\bigcap_{k=1}^{\infty} \ker \psi_k \cap \bigcap_{i=1}^{\infty} \bigcap_{j=1}^{\infty} \ker(A_j)_{v_i}$. As before, $S \subset T$. We can write T as $T = S \bigoplus_a Y$ for some subspace Y of T. Since all the common zeros of $P_j|_T$, $j \in \mathbb{N}$, are contained in S, $\sum_{j=1}^{\infty} \frac{P_j^2}{j^2||P_j||^2}$ is positive definite on Y. As S is separable we can find $(\phi_i)_{i=1}^{\infty}$ so that $\sum_{j=1}^{\infty} \frac{P_j^2}{j^2||P_j||^2} + \sum_{i=1}^{\infty} \phi_i^4$ is positive definite on T. Then

$$\sum_{j=1}^{\infty} \frac{P_j^2}{j^2||P_j||^2} + \sum_{i=1}^{\infty} \phi_i^4 + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{(A_j)_{v_i}^4}{i^2j^2||(A_j)_{v_i}||^4} + \sum_{k=1}^{\infty} \frac{\psi_k^4}{k^2||\psi_K||^4}$$

is a positive definite polynomial on E, contradicting the fact that E does not admit such a polynomial. □

Corollary 18. *Let E be a real Banach space which does not admit a positive definite 4-homogeneous polynomial. Then every $P \in P(^3E)$ is identically zero on a non-separable subspace of E.*

Theorem 4.5. *Let E be a real Banach space which does not admit a positive definite homogeneous polynomial. Then, for every polynomial P on E such that $P(0) = 0$, there is a non-separable subspace of E on which P is identically zero.*

Lemma 8. *A real Banach space E admits a positive definite 2-homogeneous (1,2)-summing polynomial if and only if there is a continuous 2-summing injection from E into a Hilbert space.*

Corollary 19. *Let E be an $L_{\infty,\lambda}$ -space for some real λ. Then every positive definite polynomial on E is (1,2)-summing.*

Note, though, that there may well not exist any positive definite polynomials on an $L_{\infty,\lambda}$ space.

We next consider the question of the existence of positive definite 2-homogeneous polynomials in case E is a $C(K)$ space. We recall that a (Borel) measure μ on a compact set K is said to be *strictly positive* if $\mu(B) > 0$ for every non-empty open subset $B \subset K$.

Corollary 20. *Let $E = C(K)$ where K is a compact Hausdorff space. Then*
(i) $C(K)$ admits a positive definite 2-homogeneous polynomial if and only if K admits a strictly positive measure.
(ii) $C(K)$ admits a positive definite 2-homogeneous nuclear polynomial if and only if there is a sequence of finite Borel measures $(\mu_n)_{n=1}^{\infty}$ on K such that $\int_K f(x)d\mu_n(x) = 0$ for all n implies $f \equiv 0$.

Theorem 4.6. *Let E be a real Banach space.*
(i) Either E admits a positive definite 2-homogeneous nuclear polynomial or every $P \in \mathcal{P}_N(^2E)$ has a non-separable subspace on which it is identically zero.
(ii) Either E admits a positive definite 2-homogeneous (1,2)-summing polynomial or every (1,2)-summing has an non-separable subspace on which it is identically zero.

Proof. (i) We reason as before, supposing that E does not admit a positive 2-homogeneous nuclear polynomial and that $P \in \mathcal{P}_N(^2E)$. Let S be a maximal subspace of E on which P is identically 0, assume that $S = \overline{\{v_i : i \in \mathbb{N}\}}$, let $T = \bigcap_{i=1}^{\infty} \ker A_{v_i}$, and write $T = S \bigoplus_a Y$. Without loss of generality, we may assume that $P|_T$ is positive definite on Y, so that $P|_T \geq 0$. Since S is separable, we can find a sequence $\{\phi_i\}_{i=1}^{\infty}$ in E' so that $\sum_{i=1}^{\infty} \phi_i^2$ is positive definite on S and nuclear on E. Hence $P + \sum_{i=1}^{\infty} \phi_i^2$ is positive definite and nuclear on T. We therefore have a continuous linear nuclear injection i of T into l_2. We can extend i to a nuclear linear map $\widetilde{i}$ from E into l_2.

Define a map $j : E \to l_2 \bigoplus_2 l_2$ by

$$j(x) = \left(\widetilde{i}(x), \sum_{i=1}^{\infty} \frac{A_{v_i}(x)}{i^2 ||A_{v_i}||} e_i \right).$$

Since j is a nuclear injection, E admits a positive definite nuclear polynomial, which is a contradiction.

(ii) The argument given above works in the (1,2)-summing case, the only significant change being an appeal to the $\prod_2$ Extension Theorem to prove the existence of a 2-summing extension mapping $\widetilde{i} : E \to L_2(I) \bigoplus_2 l_2$, for a sufficiently large index set I. □

Even if we know that an $L_{\infty,\lambda}$-space admits a positive definite (1,2)-summing polynomial, it is nevertheless possible to conclude something about the zeros of those 2-homogeneous polynomials which are not (1,2)-summing.

Theorem 4.7. *Let E be a real $L_{\infty,\lambda}$-space. Then every $P \in \mathcal{P}(^2E)$ which is not (1,2)- summing has an infinite dimensional subspace on which it is identically zero.*

Proof. Suppose $P \in \mathcal{P}(^2E)$ is not (1,2)-summing. Suppose that a maximal subspace S on which P vanishes is only finite dimensional, with basis $\{v_1, \ldots, v_n\}$. Let $T = \bigcap_{i=1}^{n} \ker A_{v_i}$, and write $T = S \bigoplus Y$, for some complemented subspace $Y \subset T$. Without loss of generality, $P|_T$ is positive definite on Y and, since S is finite dimensional, we can find $\phi_1, \ldots, \phi_n$ so that $P + \sum_{i=1}^{n} \phi_i^2$ is positive definite on T. Let π_T be the (continuous) projection of E onto T. Then $(P + \sum_{i=1}^{n} \phi_i^2) \circ \pi_T + \sum_{i=1}^{n} A_{v_i}^2$ is positive definite on E. But E is an $L_{\infty,\lambda}$-space and so by Corollary 14, $(P + \sum_{i=1}^{n} \phi_i^2) \circ \pi_T + \sum_{i=1}^{n} A_{v_i}^2$ is (1,2)-summing implying that $P|_T$ and hence P itself is (1,2)-summing, a contradiction. □

4.1. ZEROES OF REAL POLYNOMIALS ON $C(K)$ SPACES

All results of this subsection was proved in [13].

By $\mathcal{C}_H$, and $\mathcal{W}^*$, we shall denote the class formed by those Banach spaces which can be injected (i.e., there is a continuous one-to-one linear map) into a Hilbert space, and the subclass formed by those that can be injected into a separable Hilbert space, respectively. Notice that X in $\mathcal{W}^*$ is equivalent to say that X^* is weak*-separable. If Y is a closed linear subspace of X

such that Y is in $\mathcal{W}^*$ and X/Y is in $\mathcal{C}_H$, then X is in $\mathcal{C}_H$. Thus obtaining that being in $\mathcal{W}^*$ is a three-space property.

If X is not in $\mathcal{C}_H$, then every element of $\mathcal{P}(^2X)$ admits an infinite-dimensional linear subspace where it vanishes, and the following conjecture is stated:

Conjecture. *If X is a real Banach space such that $X \notin \mathcal{C}_H$, then the zero-set of every quadratic polynomial, i.e., an element of $\mathcal{P}(^2X)$, contains a non-separable linear zero subspace.*

This conjecture is proved to be correct for spaces having the Controlled Separable Projection Property (CSPP)([7]), a class which contains the Weakly Countably Determined spaces. Let us recall that X has the CSPP whenever, if (x_j) and (x_j^*) are sequences in X and X^*, respectively, there exists a norm-one projection on X with separable range containing (x_j) and such that the range of its conjugate contains (x_j^*).

Let us recall that the $(n-1)$-derivative of the polynomial $P \in \mathcal{P}(^nX)$ is given by the continuous and linear map $P^{(n-1)} : X \to \mathcal{L}_s(X^{n-1})$ such that

$$P^{(n-1)}(x) = n!\check{P}(x, \cdot, \overset{(n-1)}{\dots}, \cdot),\ x \in X,$$

where $\mathcal{L}_s(X^{n-1})$ is the space of continuous symmetric $(n-1)$-linear functionals on X and $\check{P}$ is the n-linear functional provided by the polarization formula.

Proposition 15. *Given $n \in \mathbb{N}$, if $P \in \mathcal{P}(^nc_0(\Gamma))$, then* $\ker P^{(n-1)}$ *contains an isometric copy of $c_0(\Gamma)$.*

Proof. After ([10], Exercise 1.72, p. 68), we know that $\mathcal{P}_w(^nc_0(\Gamma))$ coincides with $\mathcal{P}(^nc_0(\Gamma))$. Hence, if $P \in \mathcal{P}(^nc_0(\Gamma))$, again using ([10], Proposition 2.6, p. 88), we have that the linear map $P^{(n-1)}$ is weak-to-norm continuous on bounded sets from $c_0(\Gamma)$ into $\mathcal{L}_s(c_0(\Gamma)^{n-1})$.

For each $m \in \mathbb{N}$, we consider the set

$$\Gamma_m := \{\gamma \in \Gamma : ||P^{(n-1)}(e_\gamma)|| \geq 1/m\},$$

where e_γ stands for the unit vector in $c_0(\Gamma)$ corresponding to γ. We claim that Γ_m is finite, otherwise there would be an infinite sequence $(\gamma_j)_{j=1}^\infty$ contained in Γ_m; but, since $P^{(n-1)}$ is weak-to-norm continuous on bounded sets, and the sequence $(e_{\gamma_j})_{j=1}^\infty$ is weakly null in $c_0(\Gamma)$, this would yield

$$\lim_j ||P^{(n-1)}(e_{\gamma_j})|| = 0,$$

a contradiction. Consequently, the set

$$\Gamma_0 := \{\gamma \in \Gamma : P^{(n-1)}(e_\gamma) = 0\} \bigcup_{m=1}^{\infty} \Gamma_m$$

is countable. Thus, if E denotes the closed linear span of $\{e_\gamma : \gamma \in \Gamma \backslash \Gamma_0\}$ in $c_0(\Gamma)$, it clearly follows that E is isometric to $c_0(\Gamma)$. Besides, if $\gamma \in \Gamma \backslash \Gamma_0$, we have $P^{(n-1)}(e_\gamma) = 0$, from where we deduce that, since $P^{(n-1)}$ is linear, $E \subset \ker P^{(n-1)}$. □

In the coming result it is convenient to observe that, for $P \in \mathcal{P}(^nX)$, the linear subspace $\ker P^{(n-1)}$ is always contained in the zero-set $\ker P$.

Corollary 21. *Let Γ be an uncountable set. If $P \in \mathcal{P}(c_0(\Gamma))$, then there is a closed linear subspace E of $c_0(\Gamma)$ such that $P|_E = P(0)$ and E is isometric to $c_0(\Gamma)$.*

Corollary 22. *Every real-valued analytic function on $c_0(\Gamma)$ admits a closed linear subspace isometric to $c_0(\Gamma)$ where it has constant value.*

Notice that Proposition 15, as well as the two previous corollaries, could also be obtained from the well-known fact that any continuous polynomial on $c_0(\Gamma)$ factors through $c_0(\Gamma')$ with Γ' countable.

Proposition 16. *Let K be a compact Hausdorff topological space. The following conditions are equivalent:*
(i) $C(K)$ contains a non-separable weakly compact subset.
(ii) K does not satisfy the CCC.
(iii) $C(K)$ contains an isometric copy of $c_0(\Gamma)$, for some uncountable Γ.
(iv) There is an uncountable set Γ such that there is a one-to-one bounded linear map from $c_0(\Gamma)$ into $C(K)$.

Proof. (i) $\Rightarrow$ (ii). Let W be a weakly compact non-separable subset of $C(K)$, which we may assume to be absolutely convex. By a result of Corson, see [15], W contains a subset which is homeomorphic, in its weak topology, to the one-point compactification of an uncountable discrete set and we may thus find an uncountable subset $W_0 \subset W\backslash\{0\}$ such that every sequence of distinct elements of W_0 is weakly-null. There is clearly some $\delta > 0$ such that the set $W_1 := \{x \in W_0 : ||x|| > \delta\}$ is uncountable. For each $x \in W_1$, let $V_x := \{t \in K : |x(t)| > \delta/2\}$. Then, if $(x_j)_{j=1}^{\infty}$ is a sequence of distinct elements of W_1, it follows that $\bigcap_{j=1}^{\infty} V_{x_j} = \varnothing$, otherwise, since $x_j \to 0$ weakly, this would imply that $\lim_j x_j(t) = 0$, for all $t \in K$, in particular, if $t \in \bigcap_{j=1}^{\infty} V_{x_j}$, this would lead to a contradiction. Hence, we have an uncountable collection $\{V_x : x \in W_1\}$ of non-empty open subsets of K such that for all sequences $(V_{x_j})_{j=1}^{\infty}$ of distinct terms the intersection of its members is empty; this is a sufficient condition for K not to satisfy the CCC.

(ii)$\Rightarrow$(iii). Let $(V_\gamma)_{\gamma\in\Gamma}$ be an uncountable collection of pairwise disjoint non-empty open subsets of K. For each $\gamma \in \Gamma$, we find a function $x_\gamma \in C(K)$ such that $||x_\gamma|| = 1$ and $x_\gamma(t) = 0$, $t \in K\backslash V_\gamma$. Thus, if E denotes the closed linear span of $\{x_\gamma : \gamma \in \Gamma\}$ in $C(K)$, it is clear that E is isometric to $c_0(\Gamma)$.

(iii) $\Rightarrow$ (iv) Being obvious, we see that (iv) $\Rightarrow$ (i).

Let Γ be an uncountable set and $T : c_0(\Gamma) \to C(K)$ a one-to-one bounded linear map. Then, it is clear that the set
$$\{Te_\gamma : \gamma \in \Gamma\} \cup \{0\}$$
is weakly compact and non-separable in $C(K)$. ☐

Corollary 23. *Let K be a compact space not satisfying the CCC. For any positive integer n, every continuous n-homogeneous real-valued polynomial on $C(K)$ vanishes in an isometric copy of $c_0(\Gamma)$, for some uncountable Γ.*

Corollary 24. *If K does not satisfy the CCC, then every analytic real-valued function on $C(K)$ has constant value in an isometric copy of $c_0(\Gamma)$, for some uncountable Γ.*

l_∞/c_0 is isometric to $C(\beta\mathbb{N} \setminus \mathbb{N})$. Since it is well known that there is a family, with the continuum cardinality, of infinite subsets of $\mathbb{N}$ such that any two distinct members meet only in a finite set, it follows that $\beta\mathbb{N} \setminus \mathbb{N}$ does not have the CCC, so the next result obtains.

Corollary 25. *Every analytic real-valued function on l_∞/c_0 is constant in an isometric copy of $c_0(\Gamma)$, Γ having the continuum cardinality.*

Corollary 26. *For every positive integer n, if $P \in \mathcal{P}(^n l_\infty)$ is such that* $\ker P^{(n-1)}$ *contains c_0, then there is a closed linear subspace Z of l_∞ such that $c_0 \subset Z \subset \ker P$ and Z/c_0 is isometric to $c_0(\Gamma)$, Γ with the continuum cardinality.*

Lemma 9. *For an uncountable set Γ, the spaces $c_0(\Gamma)$, $l_p(\Gamma)$, $2 < p \leq \infty$, do not belong to the class $\mathcal{C}_H$.*

Proposition 17. *If X is a Banach space such that, for some uncountable Γ, either $c_0(\Gamma)$, or $l_p(\Gamma)$, $2 < p < \infty$, is injected into X, then X belongs to the class $\mathcal{C}'_H$.*

Proof. Let $T : c_0(\Gamma) \to X$ be a one-to-one bounded linear map (an analogous proof works for the case of $l_p(\Gamma)$ injected into X). Let $(u_j^*)_{j=1}^\infty$ be a sequence in X^* and let $Y := \bigcap_{j=1}^\infty \ker u_j^*$. As we have already seen before in similar situations, it can be seen that, for each positive integer m, and $j \geq 1$, the set

$$\Gamma_{m,j} := \{\gamma \in \Gamma : |\langle u_j^*, Te_\gamma\rangle| > 1/m\}$$

is finite, and so, for each $j \geq 1$, the set

$$\Gamma_{0,j} := \{\gamma \in \Gamma : \langle u_j^*, Te_\gamma\rangle \neq 0$$

is countable. Hence, the set $\Gamma_0 := \Gamma \setminus \bigcup_{j>0} \Gamma_{0,j}$ has the same cardinality as Γ and we have that, $Te_\gamma \in Y, \gamma \in \Gamma_0$. Denoting by E the closed linear span of $\{e_\gamma : \gamma \in \Gamma_0\}$, we obtain an isometric copy of $c_0(\Gamma)$ which is injected into Y. After the previous lemma, this implies that Y cannot be in $\mathcal{C}_H$. □

From Propositions 16 and 17 the coming result obtains.

Corollary 27. *If K does not have the CCC, then $C(K)$ belongs to the class $\mathcal{C}'_H$.*

Lemma 10. *The following statements are equivalent :*

(i) $X \in \mathcal{C}_H$.

(ii) There is a positive definite 2-homogeneous continuous polynomial on X.

(iii) If $X = C(K)$, K carries a strictly positive measure (a non-negative regular finite Borel measure which has positive value on every non-empty open subset).

Proposition 18. *Let X be in class $\mathcal{C}'_H$. Then, if $P \in \mathcal{P}(^2X)$, every maximal linear subspace contained in* $\ker P$ *is non-separable.*

Proof. Let $P \in \mathcal{P}(^2X)$. Let Z be a maximal linear subspace contained in $\ker P$, whose existence is guaranteed by Zorn's Lemma. We show that Z is non-separable. If this were not so, since the Frechet derivative $P' : X \to X^*$ is a bounded linear map, setting $Y := \{x \in X : \langle P'(z), x\rangle = 0, z \in Z\}$, it follows that $(X/Y)^* = Y^\perp = \overline{P'(Z)}^{w^*}$ is weak*-separable, i.e., Y is a countable intersection of closed hyperplanes. Hence, $X \in \mathcal{C}'_H$ implies that $Y \notin \mathcal{C}_H$. But, from the maximality of Z, it is easy to see that $Y \cap \ker P = Z$ and that P does not change sign in Y; thus, defining $Q(y + Z) := P(y), \quad y \in Y$, we obtain a quadratic polynomial in Y/Z such that either Q, or $-Q$, is positive definite. From the above lemma, this implies that $Y/Z \in \mathcal{C}_H$, and, since $Z \in W^*$, after the 3-space result it follows that $Y \in \mathcal{C}_H$, a contradiction. □

If A is a subset of the compact space K, then by $C_A(K)$ we denote the closed linear subspace of $C(K)$ formed by those functions which vanish in A.

Proposition 19. *For a compact Hausdorff topological space K, if Y is a closed linear subspace of $C(K)$ such that $C(K)/Y \in \mathcal{W}^*$ and $Y \in \mathcal{C}_H$, then $C(K) \in \mathcal{C}_H$.*

Proof. Let Y be a closed linear subspace of $C(K)$ such that it satisfies the conditions of our statement. Since $(C(K)/Y)^* = Y^\perp$ is weak*-separable, there is a sequence $(\mu_j)_{j=1}^\infty$ contained in $C(K)^*$, which, after Riesz's theorem, we identify with $M(K)$, the space of regular finite Borel measures in K, such that $Y = \bigcap_{j=1}^\infty \ker \mu_j$. In light of Jordan's decomposition theorem, there is no loss of generality in assuming that those measures are probabilities on K. We show first that K must have the Countable Chain Condition (CCC, for short). Otherwise, after Proposition 16, $C(K)$ would contain a copy (isometrically indeed) of $c_0(\Gamma)$, for some uncountable set Γ. Thus, let $S : c_0(\Gamma) \to C(K)$ be such isometry (what we need really is that S is weakly continuous and one-to-one). Then setting, for each pair of positive integers j, m,

$$\Gamma_{jm} := \{\gamma \in \Gamma : |\langle \mu_j, Se_\gamma \rangle| > 1/m\},$$

where e_γ stands for the corresponding unit vector of $c_0(\Gamma)$, it is clear that Γ_{jm} must be a finite set. Hence, the set $\Gamma_0 := \bigcup_{j,m} \Gamma_{jm}$ is countable and the closed linear span of $\{Se_\gamma : \gamma \in \Gamma \backslash \Gamma_0\}$ is contained in Y. This implies that a copy of $c_0(\Gamma \backslash \Gamma_0)$ would be injected into Y, a contradiction, since, after Lemma 9, $c_0(\Gamma \backslash \Gamma_0) \notin \mathcal{C}_H$.

Let $K_0 := \overline{\bigcup_{j=1}^\infty \mathrm{supp} \mu_j}$. Then, it is easy to see that K_0 carries a strictly positive measure and so, after Lemma 10, $C(K_0) \in \mathcal{C}_H$. Let H_1 be a Hilbert space and T_1 be a one-to-one bounded linear map from $C(K_0)$ into H_1. Having in mind that the family of cozero sets, i.e., the complements of zero-sets of elements of $C(K)$, is a base for the open sets in K, Zorn's Lemma guarantees the existence of a maximal collection of pairwise disjoint cozero sets contained in $K \backslash K_0$ (we assume that $K \backslash K_0 \neq 0$, otherwise $C(K) \in \mathcal{C}_H$). Now, the CCC forces this collection to be a countable one, so let $(V_j)_{j=1}^\infty$ represent this maximal collection. Clearly, if we set $V := \bigcup_{j=1}^\infty V_j$, then

$$V \subset K \backslash K_0 \subset \overline{V}.$$

Since V is also a cozero set, let φ be a continuous real-valued function such that $\varphi^{-1}(0) = K \backslash V$. Observing that $C_{K \backslash V}(K) \subset C_{K_0}(K) \subset Y$, we have that $C_{K \backslash V}(K) \in \mathcal{C}_H$. And so, there is a Hilbert space H_2 and a one-to-one bounded linear map T_2 from $C_{K \backslash V}(K)$ into H_2. Let H be the Hilbert space given by the product $H_1 \times H_2$. We define the map $T : C(K) \to H$ as

$$Tx := (T_1(x_{|K_0}), T_2(x\varphi)).$$

Then, it can be easily verified that T is well defined, as well as that it is linear and bounded. We see that it is one-to-one: If $Tx = 0$, then, since $\ker T_1 = \{0\}$, we have that x vanishes in K_0; also, $\ker T_2 = \{0\}$ implies that $x\varphi = 0$ and so x must also vanish in $\overline{V} \supset K \backslash K_0$; hence, $x = 0$. Therefore, $C(K) \in \mathcal{C}_H$. □

Corollary 28. *For a compact Hausdorff topological space K, the following statements are equivalent:*

(i) K does not carry a strictly positive measure.

(ii) $C(K)$ is not injected into a Hilbert space.

(iii) For every closed linear subspace Y of $C(K)$ such that $C(K)/Y \in \mathcal{W}^*$, it follows that $Y \notin \mathcal{C}_H$, i.e., $C(K) \in \mathcal{C}'_H$.

(iv) For every continuous 2-homogeneous polynomial on $C(K)$, its zero-set contains a non-separable linear subspace.

4.2. Zero Sets of Polynomials in Several Variables

All results of this subsection was proved in [4].

Let $k, n \in \mathbb{N}$ where n is odd. Let us denote by $\{e_i\}_{i=1}^k$ the canonical basis of $\mathbb{R}^k$. Given $l \in \mathbb{N}$ a partition $\mathcal{A} = (A_1, \dots, A_k)$ of $\{1, \dots l\}$ is called an *ordered partition of* $\{1, \dots l\}$ *of rank* $|\mathcal{A}| = k$.

We set $N(k,n) = \begin{pmatrix} k+n-1 \\ k-1 \end{pmatrix}$. A set $S(k,n)$ of cardinality $N(k,n)$ in $\mathbb{R}^k$ is called a *basic set of nodes* if $P|_{S(k,n)} \equiv 0$ implies $P \equiv 0$ whenever P is an n-homogeneous polynomial on $\mathbb{R}^k$.

Lemma 11. *Given $k, n \in \mathbb{N}$, there exists a set $S(k,n) = \{v^i\}_{i=1}^{N(k,n)} \subset \mathbb{R}^k$ such that $e_j \in S(k,n)$, $1 \le j \le k$, with the property that for every n-homogeneous polynomial $Q(x)$ on $\mathbb{R}^k$, if $Q(v^i) = 0$, $1 \le i \le N(k,n)$ then $Q \equiv 0$ on $\mathbb{R}^k$.*

Lemma 12. *Given $k, n \in \mathbb{N}$, $\quad k \le l$ there exist $p(k,l) \in \mathbb{N}$, $p(k,l) \le k!(\log_2(l))^k$, and a system $\{\mathcal{A}_I\}_{I=1}^{p(k,l)}$ of ordered partitions of $\{1, \dots, l\}$ of rank k, such that for every $B \subset \{1, \dots, l\}$, $|B| = k$, there exists $\mathcal{A}_I = (A_1, \dots, A_k)$ for which $B \cap A_i \ne 0$ for every $1 \le i \le k$.*

Given an (abstract) n-homogeneous nonzero polynomial $Q(x)$ on a k-dimensional Banach space X, by a suitable choice of the basis $\{\widetilde{e_1}, \dots, \widetilde{e_k}\}$ of X we can easily achieve that in the formula

$$Q(\sum_{i=1}^{k} y_i \widetilde{e_i}) = \sum_{|\alpha|=n} b^\alpha y^\alpha,$$

we have $b^{(n,0,\dots,0)} \ne 0$. Indeed, it is enough to choose a direction $\widetilde{e_1}$ in which Q is nonzero. It is easily verified that a change of variables $y_1 \to C(\sum_{i=n}^{k} x_i)$, $y_2 \to x_2, \dots, y_k \to x_k$, where C is sufficiently large, will lead to a transformed algebraic formula for the same abstract polynomial Q on X:

$$Q((x_1, \dots, x_k)) = \sum_{|\alpha|=n} a^\alpha x^\alpha,$$

in which $a^{(n,0,\dots,0)}, a^{(0,n,0,\dots,0)}, \dots, a^{(0,0,\dots,0,n)}$ are all nonzero. To summarize, we have the following.

Lemma 13. *Let $Q(x)$ be an (abstract) n-homogeneous nonzero polynomial on a k-dimensional Banach space X. Then there exists a basis $\{e_1, \dots, e_k\}$ in X such that in the formula*

$$Q(\sum_{i=1}^{k} x_i e_i) = Q((x_1, \dots, x_k)) = \sum_{|\alpha|=n} a^\alpha x^\alpha,$$

all the constants $a^{(n,0,\dots,0)}, a^{(0,n,0,\dots,0)}, \dots, a^{(0,0,\dots,0,n)}$ are nonzero.

Let us introduce the following notation. Let $A \subset \{1, 2, \dots, l\}$.

We put $P_A : \mathbb{R}^l \to \mathbb{R}^l$, $P_A(\sum_{i=1}^{k} x_i e_i) = \sum_{j \in A} x_i e_i$.

Given $v = (v_1, \ldots, v_k) \in \mathbb{R}^k$ and an ordered partition $\mathcal{A} = (A_1, \ldots, A_k)$ of $\{1, \ldots, l\}$, $|\mathcal{A}| = k$, we define $v.\mathcal{A} : \mathbb{R}^l \to \mathbb{R}^l$ as

$$v.\mathcal{A}(x) = \sum_{j=1}^{k} v_j P_{A_j}(x).$$

Theorem 4.8. *Let $n \in \mathbb{N}$, n be odd, and let $Q(x)$ be n-homogeneous polynomial on $\mathbb{R}^N$. Provided $N > k!(\log_2(N))^k \begin{pmatrix} k+n-1 \\ k-1 \end{pmatrix}$, there exists a linear subspace $X \hookrightarrow \mathbb{R}^N$, $\dim X = k$ such that $Q \equiv 0$ on X.*

Proof. By Lemma 13, we may assume that the basis $\{e_1, \ldots, e_N\}$ of $\mathbb{R}^N$ is chosen so that all the monomials in the formula for Q have nonzero coefficients. Consider the system $\{\mathcal{A}_I\}_{I=1}^{p(k,N)}$ from Lemma 12 and the set $S(k,n) = \{v^J\}_{J=1}^{N(k,n)}$ from Lemma 11. Fix a basis $e_{I,J}$, $1 \le I \le p(k,N)$, $1 \le J \le N(k,n)$ in $\mathbb{R}^{p(k,N)N(k,n)}$. Form an n-homogeneous polynomial $\widetilde{Q} : \mathbb{R}^N \to \mathbb{R}^{p(k,N)N(k,n)}$

$$\widetilde{Q}(x) = \sum_I \sum_J Q(v^J.\mathcal{A}_I(x)) e_{I,J}.$$

By assumption, $N+1 \ge p(k,N)N(k,n)$ and the mapping $\widetilde{Q}(x)$ is odd ($\widetilde{Q}(-x) = -\widetilde{Q}(x)$). By the Borsuk antipodal theorem ([16]) there exists a nonzero $x^0 = (x_1^0, \ldots, x_N^0) \in \mathbb{R}^N$, such that $\widetilde{Q}(x^0) = 0$. Denote $B = \operatorname{supp}(x^0)$.

We first claim that $|B| > k$. Indeed, otherwise there exists some $\mathcal{A}_I = (A_1, \ldots, A_k)$ such that $|B \cap A_i| \le 1$, whenever $1 \le i \le \Delta k$, and $|B \cap A_j| = 1$ for some j. Pick J such that $v^J = (0, \ldots, 0, 1, 0, \ldots, 0) = e_m \in S(k,n)$, where $\{m\} = B \cap A_j$. Clearly, $v_J.\mathcal{A}_I(x^0) = (0, \ldots, 0, x_m^0, 0, \ldots, 0)$ as all monomials in the formula for Q are nonzero, $Q(v^J.\mathcal{A}_I(x^0)) \neq 0$, a contradiction to $\widetilde{Q}(x^0) = 0$.

Thus $|B| > k$ and we can find $\mathcal{A}_I = (A_1, \ldots, A_k)$ such that $|B \cap A_i| \ge 1$, $1 \le i \le k$, which means that $x^i = P_{A_i}(x^0) \neq 0$. Next define a polynomial R on $\mathbb{R}^k$

$$R((t_1, \ldots, t_k)) = Q(\sum_{i=1}^{K} t_i x^i).$$

Since $\widetilde{Q}(x_0) = 0$, it is immediate that $R(v^J) = Q(v^J.\mathcal{A}_I(x^0)) = 0$, $1 \le J \le N(k,n)$. Thus $R \equiv 0$ on $\mathbb{R}^k$. Consequently, it suffices to choose $X = \operatorname{span}\{x^i\}_{i=1}^k$, in order to obtain $Q \equiv 0$ on X.

□

Note that in Theorem 4.8, $N = (\log_2 N)^k \to \infty$ as $N \to \infty$. In order to give an explicit formula for the asymptotic dependence of N on the values of k and n, let us note that $N \ge (k+n)^{3k}$ satisfies the requirements of Theorem 4.8, provided $k + n \ge 2^4$.

As the following corollary shows, the fact that Theorem 4.8 was stated for homogeneous polynomials is not a real restriction.

Corollary 29. *Given k and $m \in \mathbb{N}$, there is $N \in \mathbb{N}$ such that every odd polynomial $Q(x)$ of degree $2m+1$ on $\mathbb{R}^N$ vanishes on a subspace of dimension k.*

In order to motivate the last part of our note, which deals with even degree polynomials, let us recall the statement of the fundamental theorem of Dvoretzky on almost spherical sections of unit balls of finite dimensional Banach spaces.

Theorem 4.9. **(Dvoretzky)** *Let $(X, ||\cdot||)$ be an N-dimensional Banach space, $\varepsilon > 0, k \in \mathbb{N}$. There exists a function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ such that provided $k \leq \eta(\varepsilon)\log(N)$ there exists a linear operator $T : l_2^k \to X$, such that $||T||||T^{-1}|| \leq 1+\varepsilon$.*

4.3. Odd Degree Polynomials on Real Banach Spaces

All results of this subsection was proved in [5], [6].

A classical result of Birch claims that for given k, n integers, n-odd there exists some $N = N(k,n)$ such that for arbitrary n-homogeneous polynomial P on $\mathbb{R}^N$, there exists a linear subspace $Y \hookrightarrow \mathbb{R}$ of dimension at least k, where the restriction of P is identically zero (we say that Y is a null space for P).

Given $n > 1$ odd, and arbitrary real separable Banach space X (or more generally a space with w^*-separable dual X^*), we construct a n-homogeneous polynomial P with the property that for every point $0 \neq x \in X$ there exists some $k \in \mathbb{N}$ such that every null space containing x has a dimension at most k. In particular, P has no infinite dimensional null space. For a given n odd and a cardinal τ, we obtain a cardinal $N = N(\tau, n) = \exp^{n+1}\tau$ such that every n-homogeneous polynomial on a real Banach space X of density N has a null space of density τ.

In every real separable Banach space X (or more generally every real Banach space with w^*-separable dual X^*), of a n-homogeneous polynomial $P(n > 1$ arbitrary odd integer) which has no n-finite dimensional null space.

We say that the dual X^* has w^* density character $w^*\text{-dens}(X^*) = \Gamma$, if there exists a set $S \subset X^*$ of cardinality Γ, such that $\overline{S}^{w^*} = X^*$, and moreover Γ is the minimal cardinal with this property. Recall the following well-known fact.

Fact 1. Let X be a Banach space, then $w^*\text{-dens}(X^*)$ iff there exists a bounded linear injection $T : X \to l_\infty(\Gamma)$.

Theorem 4.10. *Let X be an infinite dimensional real Banach space with w^*-dens$X^* = \omega$, $n > 1$ an odd integer. Then there exists a n-homogeneous polynomial $P : X \to \mathbb{R}$ without any infinite dimensional null space. More precisely, given any $0 \neq x \in X$, $P(x) = 0$, there exists a $N \in \mathbb{N}$ such that every null space $x \in Y \hookrightarrow X$ has $\dim Y \leq N$.*

Proof. Suppose that we have already proven the statement of the theorem for $X = c_0$ and $n = 3$. Let $P : c_0 \to \mathbb{R}$ be the polynomial. Given any Banach space X with w^*-dens$X^* = \omega$, and $n = 3 + 2l$, we can construct the desired n-homogeneous polynomial $Q : X \to \mathbb{R}$ as follows. Fix any bounded linear injection $T : X \to c_0$ (put for example $T(x) = \left(\frac{f_i(x)}{i}\right)_{i=1}^{\infty}$, where $\{f_i\}_{i=1}^{\infty} \subset B_{X^*}$ is a separating set of functionals), and put $Q(x) = P \circ T(X) \cdot \left(\sum\limits_{i=1}^{\infty} \frac{1}{2^i} f_i(x)^{2l}\right)$. It is easy to verify that a linear subspace of X where Q vanishes translates via T into a linear subspace (of the same dimension) of c_0 where P vanishes, which concludes the implication. It remains to produce P on c_0. We put

$$P((x_i)) = \sum_{k=1}^{\infty} x_k \sum_{i=k+1}^{\infty} \alpha_k^i x_i^2,$$

where $\alpha_k^i > 0$, together with the auxiliary system $\tau_{k,i}^j > 0$, are chosen satisfying conditions (0)-(3) below.

(0) $\sum\limits_{k=1}^{\infty} \sum\limits_{i=k+1}^{\infty} |\alpha_k^i| < \infty$.

(1) $\frac{1}{i}\alpha_k^i > \sum\limits_{j=k+1}^{\infty} \alpha_j^i$.

(2) $\frac{1}{2i}\alpha_k^i \geq \sum\limits_{j=1}^{\infty} \tau_{k,i}^j$.

(3) $(\alpha_p^r)^2 \leq \frac{1}{16}\tau_{r,p}^q \tau_{r,q}^p$ whenever $r < p < q$.

To construct such a system of coefficients α_k^i (and auxiliary system $\tau_{k,i}^j > 0$) is rather straightforward, proceeding inductively by the infinite rows of the matrix $\{\alpha_i^k\}$. Indeed, the additional conditions always require that elements of a certain row are small enough depending on the elements of the previous rows. Note that our choice guarantees that the formula for P converges absolutely for every $x \in c_0$.

Claim. *Given any* $0 \neq x \in c_0$, $P(x) = 0$, *there exists* $N \in \mathbb{N}$ *such that for every null space* $x \in Y \hookrightarrow c_0$ *we have that* $\dim Y \leq N$.

We may assume that $||x||_\infty \leq 1$. Consider a (nonhomogeneous) 3-rd degree polynomial $R(y) = P(x+y)$.

$$R((y_i)) = \sum_{k=1}^{\infty}(x_k + y_k) \sum_{i=k+1}^{\infty} \alpha_i^k (x_i + y_i)^2.$$

Writing $R = R_0 + R_1 + R_2 + R_3$, where R_m is the m-homogeneous part of R, we obtain in particular:

$$R_2((y_i)) = \sum_{k=1}^{\infty} x_k \sum_{i=k+1}^{\infty} \alpha_k^i y_i^2 + \sum_{k=1}^{\infty} y_k \sum_{i=k+1}^{\infty} 2\alpha_k^i x_i y_i.$$

Thus $R_2((yi)) = \sum\limits_{s=1}^{\infty} \sum\limits_{l=s}^{\infty} \beta_s^l y_s y_l$, where $\beta_s^s = \sum\limits_{k=1}^{s-1} x_k \alpha_k^s$, $\beta_l^s = 2x_l \alpha_s^l$.

To prove the claim it suffices to find $N \in \mathbb{N}$, such that R_2, restricted to $Z = [e_i : i > N] \hookrightarrow c_0$ (Z has codimension N) is strictly positive outside the origin. Indeed, if so, then $R(\lambda z) = \sum\limits_{m=0}^{3} \lambda^m R_m(z)$ is a nontrivial 3-rd degree polynomial in λ, for every $z \in Z$, and in particular for every $z \in Z$ there exists some $\lambda \in \mathbb{R}$ such that $P(x + \lambda z) = R(\lambda z) \neq 0$. Now if $x \in Y \hookrightarrow c_0$ is a null space, then $Z \cap Y = \{0\}$, and so $\dim Y \leq N$, as stated.

Let us without lost of generality assume that $x_r > 0$, where $r = \min\{i : x_i \neq 0\}$. We choose $N > r$ large enough, so that the following are satisfied.

(i) $\beta_s^s = \sum\limits_{j=r}^{s-1} x_j \alpha_j^s \geq \frac{1}{2} x_r \alpha_r^s$ for every $s \geq N + 1$.

There exists a decomposition $\beta_s^s \geq \sum\limits_{i=N+1}^{\infty} \delta_s^i$, $\delta_s^i > 0$ such that

(ii) $(\alpha_p^q)^2 \leq \frac{1}{16}\delta_p^q \delta_q^p$ whenever $N < p < q$.

To see that such a choice on N is possible, we estimate using property (1), whenever $s > \frac{3}{x_r}$

$$\beta_s^s \geq x_r \alpha_r^s - \sum_{j=r+1}^{s-1} x_j \alpha_j^s \geq x_r \alpha_r^s - \sum_{j=r+1}^{s-1} \alpha_j^s \alpha > \frac{1}{2} x_r \alpha_r^s.$$

Thus $N > \frac{3}{x_r}$ guarantees that (i) is satisfied. To see (ii), for N large enough, and $s > N$, $\frac{1}{2}x_r > \frac{1}{2N} > \frac{1}{2s}$, so we have $\beta^s_s \frac{1}{2s}\alpha^s_r$. So putting $\delta^i_s = \tau^i_{r,s}$ suffices using properties (2) and (3).

The conditions are set up so that R_2 restricted to $Z = [e_i : i > N]$ satisfies

$$R_2((y_i)) \geq \sum_{p=N+1}^{\infty} \sum_{q=p+1}^{\infty} (\delta^q_p y_p^2 + \delta^p_q y_q^2 + 2\alpha^q_p x_q y_p y_q).$$

However, condition (ii) implies that

$$\begin{aligned} \delta^q_p y_p^2 + \delta^p_q y_q^2 + 2\alpha^q_p x_q y_p y_q &\geq \delta^q_p y_p^2 + \delta^p_q y_q^2 - 2\alpha^q_p |y_p y_q| \geq \frac{3}{4}(\delta^q_p y_p^2 + \delta^p_q y_q^2) \\ &+ (\frac{1}{2}\sqrt{\delta^q_p}|y_p| - \frac{1}{2}\sqrt{\delta^p_q |y_q|})^2 > \frac{1}{2}(\delta^q_p y_p^2 + \delta^p_q y_q^2). \end{aligned}$$

The last expression is clearly a positive quadratic form in variables y_p, y_q, which concludes the claim that

$$R_2((y_i)) \geq \sum p = N + 1^{\infty} \sum_{q=p+1}^{\infty} (\frac{1}{2}\delta^q_p y_p^2 + \frac{1}{2}\delta^p_q y_q^2) > 0$$

for every $0 \neq (y_i) \in Z$. □

The statement of the theorem applies to all separable Banach spaces, $l_\infty, C(K)$, where K is separable (not necessarily metrizable). It is inherited by the subspaces, so since $l_1(c) \hookrightarrow l_\infty$, it applies also to l_1.

Our objective now is to obtain some estimate on the size of $card(\Gamma)$, such that every n-homogeneous odd polynomial on $l_1(\Gamma)$ has large null sets. Given and ordinal Γ, we say that a polynomial $P : l_1(\Gamma) \to \mathbb{R}$ is subsymmetric if $P\left(\sum_{i=1}^{l} x_i e_{\gamma_i}\right) = P\left(\sum_{i=1}^{l} x_i e_{\beta_i}\right)$ whenever we have $\gamma_1 < \gamma_2 < \ldots < \gamma_l$, $\beta_1 < \ldots < \beta_l$, for arbitrary $x_i \in \mathbb{R}$.

Lemma 14. *Let $P : l_1(\Gamma) \to \mathbb{R}$ be a subsymmetric n-homogeneous polynomial, n odd. Then P has a null set of density Γ.*

Denote by $\exp \alpha = 2^\alpha$, $\exp^{n+1} \alpha = \exp(\exp^n \alpha)$, where α is a cardinal. For a set S, let $[S]^n = \{X \subset S : \mathrm{card} X = n\}$. We will use the following result, which in the language of partition relations claims that $(exp^{n-1}\alpha)^+ \to (\alpha^+)\alpha^n_\alpha$.

Theorem 4.11. (Erdos, Rado) *Let α be an infinite cardinal, $n \in \mathbb{N}, \kappa = (exp^{n-1}\alpha)^+$ and $\{G_\gamma\}_{\gamma<\alpha}$ be a partition of $[\kappa]^n$. Then there exist $M \subset \kappa, \quad cardM = \alpha^+$ and $[M]^n \subset G_\gamma$ for some $\gamma < \alpha$.*

Proof. Let P be an n-homogeneous polynomial, suppose Γ is an ordinal. We partition the set $[\Gamma]^n$ using continuum many sets $\{G_{\{a_{i_1,\ldots,i_n}:1\leq i_1 \leq \ldots \leq i_n \leq n\}} : a_{i_1,\ldots,i_n} \in \mathbb{R}\}$ as follows.

We put $[\gamma_1, \ldots, \gamma_n] \in G_{\{a_{i_1,\ldots,i_n}:1\leq i_1\leq\ldots\leq i_n\leq n\}}$ iff $\{a_{i_1,\ldots,i_n} : 1 \leq i_1 \leq \ldots \leq i_n \leq n\}$ coincides with the set of coefficients of P, when restricted to the n-dimensional space with coordinate vectors $e_{\beta_1}, \ldots, e_{\beta_n}$ where $\{\beta_i\}$ is an increasingly reordered set $\{\gamma_i\}$ (in the order coming from Γ). Applying the Erdos-Rado theorem 4.11 yields a subset $S \subset \Gamma$ of the desired cardinality, such that the restriction of P to $l_1(S)$ is a subsymmetric polynomial. □

Theorem 4.12. *Suppose $card\Gamma \leq \exp^n \alpha$, n odd. Then every n-homogeneous polynomial on $l_1(\Gamma)$ has a null space of density at least α^+.*

Theorem 4.13. *Let X be a real Banach space of dens$(X) \geq \exp^{n+1} \alpha$, where α is a cardinal, n odd integer. Then every n-homogeneous polynomial on X has a null space of density at least α^+.*

Proof. Let $\Gamma = \exp^n \alpha$. We construct a continuous injection $T : l_1(\Gamma) \to X$ inductively as follows. Having chosen $T(e_i) \in B_X$ for all $i < \beta < \Gamma$ together with functionals $f_i \in B_{X^*}$, $f_i(T(e_i)) \geq \frac{1}{2}$, we choose $T(e_\beta) \in \bigcap_{i<\beta} \ker f_i$. The last set is nonempty, since card$X \leq 2^{w^*-\mathrm{dens}X^*}$, so w^*-dens$X^* \geq \exp^n \alpha$ and we can continue the inductive process. Now it remains to note that $P \circ T$ is an n-homogeneous polynomial on $l_1(\Gamma)$, its null subspaces carry right into X, and the previous theorem applies. □

Proposition 20. *Let Γ be an infinite cardinal, $P : c_0(\Gamma) \to \mathbb{R}$ be an arbitrary continuous polynomial. Then P has a null space of separable codimension in $c_0(\Gamma)$.*

Proof. Since P is wsc, it mapps in particular ω-null sequences to sequences convergent to $0 \in \mathbb{R}$. Using a standard argument we see that P depends only on a countable set of coordinates $S \subset \Gamma$, and so P restricted to $\Gamma \backslash S$ is identically zero. □

A similar proof based on wsc property for polynomials of degree less than p on l_p spaces gives.

Proposition 21. *Let Γ be an infinite cardinal, $P : l_p(\Gamma) \to \mathbb{R}$ be an arbitrary continuous polynomial of degree less than p. Then P has a null space of separable codimension in $l_p(\Gamma)$.*

In order to investigate polynomials of degree higher than p on $l_p(\Gamma)$ spaces, we need the following lemma.

Lemma 15. *Let P be a polynomial of n-th degree on $l_p(\Gamma)$, $\quad \Gamma > w, \quad n < 2[p]$. Then there exists a subset $\Gamma' \subset \Gamma$, linearly ordered, such that the restriction of P to Γ' has the form*

$$P((x_i)) = \sum_{j \in \Gamma', [p] \leq m \leq n} \quad \sum_{i_1 \leq \ldots \leq i_l \leq j} \alpha^m_{i_1,\ldots,i_l,j} x_{i_i} \ldots x_{i_l} x_j^m$$

The previous proposition may be further generalized to arbitrary degree polynomial. The resulting formula will contain only those mixed terms whose last power is of degree at least $[p]$.

Proposition 22. *Let P be a n-homogeneous polynomial on $l_p(w_1^+)$, $\quad n < 2[p]$. Then P has an infinite dimensional (block) null space.*

Proof. Consider the P in the above form. Since for every j, the set of nonzero $a^m_{i_1,\ldots,i_l,j}$ is at most countable. We proceed inductively as follows. Pick the first ω_1 elements of $\Gamma = \omega_1^+$. It follows that there is some $k_0 \in \Gamma$, and a set Γ_1, $\min \Gamma_1 > k_0$, of cardinality ω_1^+ such that $a^m_{i_1,\ldots,i_l,j} = 0$, whenever $k_0 \in \{i_1, \ldots, i_l\}$, for all $j \in \Gamma_1$. Since ω_1^+ is a regular cardinal, we can in the next step choose the initial ω_1-interval of Γ_1, and k_1 in there, such that for some $\Gamma_2 \subset \Gamma_1$, min $\Gamma_2 > k_1$ of cardinality ω_1^+ we have that $a^m_{i_1,\ldots,i_l,j} = 0$, whenever $k_1 \in \{i_1, \ldots, i_l\}$, for all $j \in \Gamma_2$.

We proceed inductively along ω. The final set $\{k_j\}_{j=0}^\infty$ clearly defines a splitting of P restricted to this index set. □

Proposition 23. *Let P be a 3rd degree polynomial on $l_2(w_1)$. Then P has an infinite dimensional null (block) space.*

Proof. Without lost of generality, P has the formula

$$P((x_i)) = \sum_{j<\omega_1}\sum_{i\le j} a_{i,j}x_ix_j^2.$$

We are going to construct a block sequence $\{u_k\}_{k=1}^{\infty}$ inductively as follows. First step. If there exists some i such that $\Gamma_1 = \{j : i < j, a_{i,j} = 0\}$ is uncountable, then we choose $u_1 = e_i$. Clearly, P restricted to $[u_1, e_i : i \in \Gamma_1]$ splits with respect to the decomposition $\{i\}, \Gamma_1$.

Otherwise, for every i there exists $\varepsilon_i > 0$ such that $\Delta_i = \{j : j > l, |a_{l,j}| > \varepsilon_i\}$ is uncountable. Fix $i = 1$ and still using the previous assumption, pick an $l > 1$ such that the set $\Gamma_1 = \{j : j \in \Delta_1, j > i, |a_{i,j}| < \frac{\varepsilon_1}{2}\}$ is uncountable. Here we are using the property of the ground space l_2, namely if such a choice were not possible, we would have some j for which the set $\{i : i < j, |a_{i,j}| \ge \frac{\varepsilon_1}{2}\}$ is infinite. This is a contradiction with the continuity of the linear term in the shifted polynomial $Q(x) = P(e_j + x)$. Assume, without lost of generality, that there exists some $\delta > 0, \quad a = \varepsilon_1 > a - 3\delta > \frac{\varepsilon_1}{2} > b > b - 3\delta > c \ge 0$, and a disjoint decomposition of Γ_1 into uncountable subsets Γ_1^1, Γ_1^2 such that $|a_{1,j} - a| < \delta$ for all $j \in \Gamma_1$, $|a_{l,j} - b| < \delta$ for all $j \in \Gamma_1^1$ and $|a_{l,j} - c| < \delta$ for all $j \in \Gamma_1^2$. Put $u_1 = e_l - \frac{b=c}{2a}e_1$. Consider now the polynomial P restricted to the subspace generated by the basic long sequence $\{e_i^1 : i < \omega_1\} = \{u_1, e_j : j \in \Gamma_1\}$. Its formula has the canonical form $P((x_i)) = \sum\limits_{j<\omega_1}\sum\limits_{i\le j} a_{i,j}^1 x_ix_j^2$, where moreover $|a_{1,i}^1| > \delta$ for all $i > 1$, and both sets $A = \{i : i > 1, a_{1,i}^1 > \delta\}$ and $B = \{i : i > 1, a_{1,i}^1 < -\delta\}$ are uncountable. Blocking once more, this time using a bijection $\phi : A \to B$ and suitable coefficients $c_i, i \in A$ we obtain the disjoint blocks $v_i = e_i + c_ie_{\phi(i)}$, $i \in A$, such that in the restriction of P to $[e_1^1, v_i]$ splits with respect to e_1 and $[v_i]$. The inductive step consists of repeating the previous argument, for the polynomial P restricted to the last index set defining the previous splitting. This leads to a sequence $\{u_k\}_{k=1}^{\infty}$, where each u_k lies in the block subsequent to blocks containing $u_i, i < k$, and defining a splitting of P. Thus P splits with respect to disjoint block vectors $\{u_k\}_{k=1}^{\infty}$, and the result follows. □

Remark 2. The assumption that τ is uncountable cannot be dropped. Indeed, consider the subspace of l_p generated by vectors $v_n = \sum\limits_{i=k_n}^{\infty} a_i^n e_i$ for some fast decreasing sequence $a_i^n \searrow 0$, and fast increasing $k_n \to \infty$. We have $\{v_n\} \sim \{e_n\}$ the canonical basis. The coordinates of $v_j(i), j \le n$ in the intervals $i \in [k_n, k_{n+1})$ are chosen so that for every pair of nonzero vectors $x = \sum\limits_{j=1}^{n} b_jv_j$, $y = \sum\limits_{j=1}^{n} c_jv_j$ there exists some $i \in [k_n, k_{n+1})$ for which $x(i), y(i) \neq 0$. This can be obtained by a simple compactness argument. It follows, that $[v_n : n \in \mathbb{N}]$ contains no two nonzero disjoint blocks.

Given $n - 2 < p \le n$, where n is odd, we define a polynomial operator $Q_p : l_p(c) \to l_1(c)$ by $Q((x_i)) = (x_i^n)$. Clearly, Q is n-homogeneous and injective. Let P be the 3-homogeneous polynomial on $l_1(c)$ without any infinite dimensional null space.

Lemma 16. *$R = P \circ Q$ is a $3n$-homogeneous polynomial on $l_p(c)$, which has no infinite dimensional block null space. In particular, it has no nonseparable null space. Moreover, for every $l \ge 4n + 1$ odd, there exists an l-homogeneous polynomial on $l_p(c)$ without a nonseparable null space.*

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Зроблено огляд основних результатів про лінійні підпростори у ядрах поліномів на дійсних та комплексних банахових просторах.

Ключові слова: поліноми, лінійні підпростори, ядра поліномів на банахових просторах.